CMPSCI 240: “Reasoning Under Uncertainty”

Lecture 23

Prof. Hanna Wallach
wallach@cs.umass.edu

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Reminders

- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- Seventh homework is due TOMORROW

Grades

1. Add discussion section scores, divide by 14, multiply by 10
2. Add homework scores, divide by 250, multiply by 30
3. Divide midterm score by 100, multiply by 30
4. Add 1–3 to obtain your score (max. possible is 70)
5. If your score is 40 or less, you're in danger of getting a D

Recap

Last Time: Markov Chains

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$$p_{ij} = P(X_t = j \mid X_{t-1} = i)$$

- Transition probabilities out of any state sum to one

Last Time: n-Step Transition Probabilities

- If we know the initial probability of each state $P(X_0 = i)$ and the transition probabilities, then we can compute the n-step transition probability $P(X_n = j)$ using the recursive formula:

$$P(X_n = j) = \sum_i P(X_{n-1} = i)\, p_{ij}$$

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$$P(X_n = j) = \sum_i P(X_{n-1} = i)\, p_{ij}$$

- This is just the law of total probability!

Matrices and Vectors

Transition Probability Matrices

- We can define a stochastic transition probability matrix:

$$A = \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{pmatrix}$$

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- Each entry is between 0 and 1: $0 \leq p_{ij} \leq 1$
- Each row sums to 1: $\sum_j p_{ij} = 1$

State Vectors

- We can also define a vector $v^{(t)}$ that represents the probability of the system being in each state at time t:

$$v^{(t)} = \langle P(X_t = 0), P(X_t = 1), P(X_t = 2), P(X_t = 3) \rangle$$

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- Matrix multiplication: compute any $v^{(t)}$ from $v^{(0)}$ and A
- More concise way of representing using the law of total probability to compute n-step transition probabilities

n-Step Transition Probabilities

- Using the law of total probability we saw that:

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- $P(X_n = j) = v_j^{(n)}$ and $P(X_{n-1} = i) = v_i^{(n-1)}$ and $p_{ij} = A_{ij}$

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$$v_j^{(n)} = \sum_i v_i^{(n-1)} A_{ij}$$

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$$P(X_n = j) = \sum_i P(X_{t-1} = i)\, p_{ij}$$

- $P(X_n = j) = v_j^{(n)}$ and $P(X_{n-1} = i) = v_i^{(n-1)}$ and $p_{ij} = A_{ij}$

$$v_j^{(n)} = \sum_i v_i^{(n-1)} A_{ij} \quad \Longrightarrow \quad v^{(n)} = v^{(0)} A^n$$

Examples of n-Step Transition Probabilities

- e.g., my router can be either online or offline. If it is online it will remain online the next day with probability 0.8. If it is offline it will remain offline the next day with probability 0.4. If it is initially online, what is the probability that it will be online after 4 days? What is the probability it will be offline?

Examples of n-Step Transition Probabilities

- e.g., consider a line at the Apple store. Every minute, the server serves a customer with probability $1/2$. Every minute someone joins the line with probability $1/2$ if the line has length 0; with probability $2/3$ if the line has length 1; with probability $1/3$ if the line has length 2; with probability 0 if the line has length 3. What is the transition probability matrix? If there is initially one customer in line, what is $v^{(8)}$? What is $v^{(\infty)}$? What if there are initially three customers in line?

Examples of *n*-Step Transition Probabilities

$$v^{(0)} = \langle 0.000, 1.000, 0.000, 0.000 \rangle$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle \\
v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle \\
v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle \\
v^{(3)} &= \langle 0.158, 0.416, 0.342, 0.084 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$
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v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle \\
v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle \\
v^{(3)} &= \langle 0.158, 0.416, 0.342, 0.084 \rangle \\
v^{(4)} &= \langle 0.148, 0.401, 0.352, 0.099 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$
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v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle \\
v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle \\
v^{(3)} &= \langle 0.158, 0.416, 0.342, 0.084 \rangle \\
v^{(4)} &= \langle 0.148, 0.401, 0.352, 0.099 \rangle \\
v^{(5)} &= \langle 0.142, 0.391, 0.359, 0.109 \rangle
\end{aligned}
$$

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v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle \\
v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle \\
v^{(3)} &= \langle 0.158, 0.416, 0.342, 0.084 \rangle \\
v^{(4)} &= \langle 0.148, 0.401, 0.352, 0.099 \rangle \\
v^{(5)} &= \langle 0.142, 0.391, 0.359, 0.109 \rangle \\
v^{(6)} &= \langle 0.136, 0.386, 0.364, 0.114 \rangle
\end{aligned}
$$

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v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle \\
v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle \\
v^{(3)} &= \langle 0.158, 0.416, 0.342, 0.084 \rangle \\
v^{(4)} &= \langle 0.148, 0.401, 0.352, 0.099 \rangle \\
v^{(5)} &= \langle 0.142, 0.391, 0.359, 0.109 \rangle \\
v^{(6)} &= \langle 0.136, 0.386, 0.364, 0.114 \rangle \\
v^{(7)} &= \langle 0.133, 0.382, 0.368, 0.118 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

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v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
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v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle \\
v^{(3)} &= \langle 0.158, 0.416, 0.342, 0.084 \rangle \\
v^{(4)} &= \langle 0.148, 0.401, 0.352, 0.099 \rangle \\
v^{(5)} &= \langle 0.142, 0.391, 0.359, 0.109 \rangle \\
v^{(6)} &= \langle 0.136, 0.386, 0.364, 0.114 \rangle \\
v^{(7)} &= \langle 0.133, 0.382, 0.368, 0.118 \rangle \\
v^{(8)} &= \langle 0.130, 0.380, 0.370, 0.120 \rangle \\
\vdots \quad & \vdots \quad \vdots
\end{aligned}
$$

Examples of n-Step Transition Probabilities

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v^{(0)} &= \langle 0.000, 1.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.167, 0.500, 0.333, 0.000 \rangle \\
v^{(2)} &= \langle 0.167, 0.444, 0.333, 0.056 \rangle \\
v^{(3)} &= \langle 0.158, 0.416, 0.342, 0.084 \rangle \\
v^{(4)} &= \langle 0.148, 0.401, 0.352, 0.099 \rangle \\
v^{(5)} &= \langle 0.142, 0.391, 0.359, 0.109 \rangle \\
v^{(6)} &= \langle 0.136, 0.386, 0.364, 0.114 \rangle \\
v^{(7)} &= \langle 0.133, 0.382, 0.368, 0.118 \rangle \\
v^{(8)} &= \langle 0.130, 0.380, 0.370, 0.120 \rangle \\
\vdots \;\; &\vdots \;\; \vdots \\
v^{(\infty)} &= \langle 0.125, 0.375, 0.375, 0.125 \rangle
\end{aligned}$$

Examples of n-Step Transition Probabilities

$$v^{(0)} = \langle 0.000, 0.000, 0.000, 1.000 \rangle$$

Examples of n-Step Transition Probabilities

$$\begin{aligned} v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\ v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \end{aligned}$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle \\
v^{(3)} &= \langle 0.028, 0.250, 0.472, 0.251 \rangle
\end{aligned}$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle \\
v^{(3)} &= \langle 0.028, 0.250, 0.472, 0.251 \rangle \\
v^{(4)} &= \langle 0.056, 0.296, 0.404, 0.204 \rangle
\end{aligned}
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Examples of n-Step Transition Probabilities

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\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle \\
v^{(3)} &= \langle 0.028, 0.250, 0.472, 0.251 \rangle \\
v^{(4)} &= \langle 0.056, 0.296, 0.404, 0.204 \rangle \\
v^{(5)} &= \langle 0.078, 0.324, 0.423, 0.177 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle \\
v^{(3)} &= \langle 0.028, 0.250, 0.472, 0.251 \rangle \\
v^{(4)} &= \langle 0.056, 0.296, 0.404, 0.204 \rangle \\
v^{(5)} &= \langle 0.078, 0.324, 0.423, 0.177 \rangle \\
v^{(6)} &= \langle 0.093, 0.341, 0.407, 0.159 \rangle
\end{aligned}
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Examples of n-Step Transition Probabilities

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\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle \\
v^{(3)} &= \langle 0.028, 0.250, 0.472, 0.251 \rangle \\
v^{(4)} &= \langle 0.056, 0.296, 0.404, 0.204 \rangle \\
v^{(5)} &= \langle 0.078, 0.324, 0.423, 0.177 \rangle \\
v^{(6)} &= \langle 0.093, 0.341, 0.407, 0.159 \rangle \\
v^{(7)} &= \langle 0.104, 0.353, 0.397, 0.148 \rangle
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle \\
v^{(3)} &= \langle 0.028, 0.250, 0.472, 0.251 \rangle \\
v^{(4)} &= \langle 0.056, 0.296, 0.404, 0.204 \rangle \\
v^{(5)} &= \langle 0.078, 0.324, 0.423, 0.177 \rangle \\
v^{(6)} &= \langle 0.093, 0.341, 0.407, 0.159 \rangle \\
v^{(7)} &= \langle 0.104, 0.353, 0.397, 0.148 \rangle \\
v^{(8)} &= \langle 0.111, 0.360, 0.389, 0.140 \rangle \\
\vdots \quad & \vdots \quad \vdots
\end{aligned}
$$

Examples of n-Step Transition Probabilities

$$
\begin{aligned}
v^{(0)} &= \langle 0.000, 0.000, 0.000, 1.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.000, 0.500, 0.500 \rangle \\
v^{(2)} &= \langle 0.000, 0.167, 0.500, 0.333 \rangle \\
v^{(3)} &= \langle 0.028, 0.250, 0.472, 0.251 \rangle \\
v^{(4)} &= \langle 0.056, 0.296, 0.404, 0.204 \rangle \\
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v^{(7)} &= \langle 0.104, 0.353, 0.397, 0.148 \rangle \\
v^{(8)} &= \langle 0.111, 0.360, 0.389, 0.140 \rangle \\
\vdots \quad & \vdots \quad \vdots \\
v^{(\infty)} &= \langle 0.125, 0.375, 0.375, 0.125 \rangle
\end{aligned}
$$

Steady State

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- If $vA = v$ then v is a steady state distribution

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- "Most" Markov chains have a unique steady state distribution that is approached by successive time steps (applications of transition matrix A) from any starting distribution

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- If $vA = v$ then v is a steady state distribution
- "Most" Markov chains have a unique steady state distribution that is approached by successive time steps (applications of transition matrix A) from any starting distribution
- $vA = v$ defines a set of $n+1$ simultaneous equations

Examples of Steady State Distributions

- e.g., suppose you have a Markov chain with 3 states. You know that the transition probabilities are $p_{12} = 0.5$, $p_{13} = 0.5$, $p_{21} = 0.75$, $p_{23} = 0.25$, $p_{32} = 0.75$, and $p_{33} = 0.25$. Find a steady state distribution for this Markov chain.

For Next Time

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