CMPSCI 240
Reasoning Under Uncertainty
Homework 5

Prof. Hanna Wallach

Assigned: March 02, 2012
Due: March 09, 2012

Question 1: (20 points) Two children, Xavier and Yolanda, are given an allowance. However, the amounts of money they receive are determined in a peculiar way: their parents roll a fair n-sided die and give Xavier the amount of money indicated by the die. Therefore, Xavier's allowance can be modeled by a random variable X, which can take on values $1, \ldots, n$. After Xavier's allowance (i.e., the value x of X) has been determined, they roll a second fair die, this time with x sides. The resultant number can therefore be modeled by another random variable Z, which can take on values $1, \ldots, x$. Yolanda's allowance is twice the value of Z. Therefore, Yolanda's allowance can be modeled by the random variable $Y = 2\,Z$.

(a) Find $p_X(x)$ and $\mathbb{E}[X]$.

(b) Find $p_{Z|X}(z \mid x)$ and $\mathbb{E}[Z \mid X = x]$.

(c) Use your answer to (b) to find $\mathbb{E}[Y \mid X = x]$.

(d) Use your answers to (a) and (c) to find $\mathbb{E}[Y]$.

(e) On average, does one child receive more money than the other? Why?

(f) Find $p_{X,Z}(x, z)$.

(g) Use your answer to (f) to find $\mathbb{E}[XY]$.

(h) Find $\text{cov}(X, Y)$.

Question 2: (8 points) A stream of data is sent over a noisy channel, where the probability that a bit is accidentally flipped is 0.001. You receive a block of 1000 bits and model number of flipped bits as a random variable X.

(a) What values can X take on? What is the PMF of X?

(b) Find the expected value of X.

(c) Use the Markov inequality to find an upper bound on the probability that the block has 5 or more flipped bits.

(d) Now calculate the actual probability that the block has 5 or more flipped bits. What can you conclude about the bound provided by the Markov inequality from this probability and your answer to (c)?

Question 3: (4 points) The number of cars to arrive at an intersection in an hour can be modeled by a Poisson random variable X with $\lambda = 100$.

(a) Find $\mathbb{E}[X]$ and $\text{var}(X)$.

(b) Use the Chebyshev inequality to find a lower bound on the probability that the number of cars to arrive at the intersection in an hour is between 70 and 130 inclusive. Hint: you may wish to make use of the fact that $P(-c \leq X - \mu \leq c) = P(|X - \mu| \leq c)$, where $\mu = \mathbb{E}[X]$.

Question 4: (6 points) This question is about the 7/4 Hamming code—i.e., a Hamming code with 4 "data" bits $s_1 s_2 s_3 s_4$ and 3 "parity" bits $t_5 t_6 t_7$.

(a) Determine the 16 possible 7-bit code words.

(b) Decode 1101011, 0110110, 0100111, and 1111111.

(c) What is the probability of an undetected error if the probability of a single bit being flipped is p. Hint: recall that for the 7/4 Hamming code, $P(\text{undetected error}) = P(\text{2 or more flipped bits})$.

Question 5: (12 points) Random variable X has mean $\mathbb{E}[X] = 5$, while random variable Y has mean $\mathbb{E}[Y] = 7$. Both X and Y have variance 2.4.

(a) Use the definition of variance to find the values of $\mathbb{E}[X^2]$ and $\mathbb{E}[Y^2]$.

(b) Use the definition of variance, along with your answer to (a) and the fact that $\text{var}(X+Y) = 8$, to find the value of $\mathbb{E}[XY]$.

(c) Show (algebraically) that for any scalars a, b, c, and d

$$\text{cov}(aX+bY, cX+dY) = ac\,\text{var}(X) + (ad+bc)\,\text{cov}(X,Y) + bd\,\text{var}(Y).$$

(d) Find the value of $\text{cov}(X+Y, X+1.2\,Y)$.

(e) Find the value of $\rho(X+Y, X+1.2\,Y)$. Hint: you may wish to make use of the fact that $\text{var}(aX+bY) = \text{cov}(aX+bY, aX+bY)$.