CMPSCI 240
Reasoning Under Uncertainty
Homework 3

Prof. Hanna Wallach

Assigned: February 17, 2012
Due: February 24, 2012

Question 1: (15 points) Prof. Wallach shops for probability books for K hours, where K is a random variable that is equally likely to take on the values 1, 2, 3, or 4. The number of books N that she buys is random and depends on how long she shops according to the conditional PMF

$$p_{N|K}(n \mid k) = \frac{1}{k} \quad \text{for } n = 1, \ldots, k.$$

(a) Find the joint PMF of K and N.

(b) Find the marginal PMF of N.

(c) Find the conditional PMF of K given that $N = 2$.

(d) Find the conditional mean and variance of K, given that Prof. Wallach bought at least 2 but no more than 3 books.

(e) The cost of each book is a random variable with mean \$30. What is the expected value of her total expenditure? (Hint: Condition on the events $N = 1, \ldots, N = 4$ and use the total expectation theorem.)

Question 2: (15 points) Suppose that X and Y are independent, identically-distributed geometric random variables with the following PMF:

$$p_X(k) = p_Y(k) = p\,(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots,$$

where $0 < p \leq 1$. Show that for any integer $n \geq 2$, the conditional PMF

$$p_{X|X+Y=n}(k) = P(\{X = k\} \mid X + Y = n)$$

is uniform.

Question 3: (15 points) Phil plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, independent of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y be the number of weeks that he won.

(a) What is the probability that he played the lottery on any particular week, given that he did not win on that week?

(b) Find the conditional PMF $p_{Y|X}(y \mid x)$.

(c) Find the joint PMF $p_{X,Y}(x, y)$.

(d) Find the marginal PMF $p_Y(y)$. (Hint: One possibility is to start with your answer to part (c), but the algebra can be messy; however, if you think about whether Y can be represented in terms of a sum of random variables, you may instead be able to guess the answer.)

(e) Use the preceding answers to algebraically find $p_{X|Y}(x \mid y)$.

(f) Rederive your answer to part (e) by thinking as follows: for each of the $n - Y$ weeks that Phil did not win, the answer to part (a) should tell you something about the PMF of X given $Y = y$.

Question 4: (5 points) The UMass football team wins any single game with probability p and loses with probability $1 - p$. Its performance in each game is independent of its performance in other games. Let L_1 be the number of losses before its first win, and let L_2 be the number of losses after its first win and before its second win. Find the joint PMF of L_1 and L_2.