CMPSCI 105 – Lecture #2
Numbers and the Computer

Definition: BIT

- Binary Digit
- Smallest possible unit of information
- Two values only: 0 or 1
- Represent a single Yes or No question
- Can encode any two-valued system
  - Yes/No, True/False, Up/Down, On/Off, In/Out, etc.
- Easy to build hardware to encode bits.

Bits and Patterns

- 1 Bit gives $2^1 = 2$ patterns: 0 or 1
- 2 Bits gives $2^2 = 4$ patterns: 00, 01, 10, 11
- 3 Bits gives $2^3 = 8$ patterns: 000, 001, 010, 011, 100, 101, 110, 111
- Each new bit doubles the number of patterns
- Therefore: N Bits gives 2^N Distinct patterns.

What About 8 Bits?

- 00000000 = 0
- 00000001 = 1
- 00000010 = 2
- 00000011 = 3
- 00000100 = 4
- 00000101 = 5
- …
- 11111110 = 254
- 11111111 = 255

Definition: Byte

- Packet of 8 Bits (French word is “octet”)
- Typical unit of computer memory / storage
- Used to represent one standard character
- Values range from 00000000 … 11111111
- 2^8=256 Distinct patterns
- Can encode any integer between 0 and 255

Unsigned Integers

- Pick storage size of N bits (8, 16, 32, 64, etc.)…
- …therefore 2^N distinct patterns are available.
- Smallest value is all zeroes (decimal value 0),
- Largest value is therefore 2^N-1.
- Results less than zero are “underflow” errors,
- Results greater than max are “overflow” errors
- Each computer architecture has a fixed N.

Signed Integers

- Pick N, there are still 2^N patterns.
- Consider half the patterns to be negative.
  - Half of $2^N = 2^N/2 = 2^{N-1}$.
- Remaining patterns are zero and above.
- Signed range is therefore $-2^{N-1} \ldots +2^{N-1}-1$.
  - Zero is considered positive.

Example for N=8

- $2^8 = 256$ patterns
- Unsigned Range
  - Minimum: 0
  - Maximum: $2^8-1 = 255$
- Signed Range
  - Minimum: $-2^{8-1} = -2^7 = -128$
  - Maximum: $+2^{8-1}-1 = +2^7-1 = +128-1 = +127$

Example for N=16

- $2^{16} = 65536$ patterns
- Unsigned Range
  - Minimum: 0
  - Maximum: $2^{16}-1 = 65535$
- Signed Range
  - Minimum: $-2^{16-1} = -2^{15} = -32768$
  - Maximum: $+2^{16-1}-1 = +2^{15}-1 = +32768-1 = +32767$

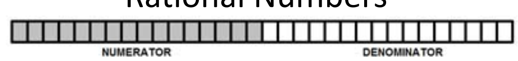
Example for N=32

- $2^{32} = 4{,}294{,}967{,}296$ patterns
- Unsigned Range
  - Minimum: 0
  - Maximum: $2^{32}-1 = 4{,}294{,}967{,}295$
- Signed Range
  - Minimum: $-2^{32-1} = -2^{31} = -2{,}147{,}483{,}648$
  - Maximum: $+2^{32-1}-1 = +2^{31}-1 = +2{,}147{,}483{,}647$
  - Nine (and a little more) significant digits

What about Real numbers?

- Approaches:
  - Rational
  - Fixed-Point
  - Floating-Point
- All require re-interpreting how bits are used.

Rational Numbers

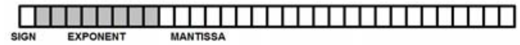


- For N bits, divide into two $^N/_2$ bit sections:
  - First section is numerator
  - Second section is denominator
- Numbers like $^1/_3$, $^1/_2$, $^3/_7$, $^1/_{10}$, $^{355}/_{113}$ are easy
- Reduce to lowest form (e.g., $^2/_4$ goes to $^1/_2$)
- Many redundant patterns:
  - low information density
  - Not efficient use of bits

Fixed-Point Numbers

- Set virtual decimal point to middle of bits:
  – Half the bits are integer
  – Half the bits are fraction
- All bit patterns are useful
- Easy to add, subtract, multiply, divide in binary
- Trades off range of values for fraction support.
  – For N=16, max signed value is only +127.99609375
- Still not an efficient use of bits

Floating-Point Numbers



- Binary version of Scientific Notation
  – Decimal: $+3.4024\times10^{15}$
  – Binary: $+1.00101001\times2^{1001}$
- Use one bit for *sign* (0=plus, 1=minus)
- Use some of the N bits for *exponent*
- Use remaining bits for *mantissa* (significand)
- Trades off <u>precision</u> for <u>dynamic range</u>

Floating-Point Precision

- Single Precision
  – N=32 bits (1 sign, 8 exponent, 23 mantissa)
  – Dynamic Range: $\pm10^{\pm38}$
  – Significant Figures: 5-6 Decimal Digits
  – (Remember 32 bit integers have about 9 sig. figs.)
- Double Precision (Used by Excel)
  – N=64 bits (1 sign, 11 exponent, 52 mantissa)
  – Dynamic Range $\pm10^{\pm308}$
  – Significant Figures: 15-16 Decimal Digits

But long fractions get rounded off:

- Expected loss of precision:
  – Numbers with naturally long but finite fractions,
  – Rationals that repeat forever (⅓ = 0.333333333…),
  – Irrationals (e, π, ϕ, √2, √3, √5, etc.).
- Unexpected loss of precision: Well-behaved decimal fractions that are ill-behaved in binary ($^1/_{10}$ = 0.00011001100110011…)

Aside: Proof that √2 is Irrational

- √2 = 1.414213562…
- Remember:
  – Even × Even = Even (4 × 6 = 24)
  – Even × Odd = Even (4 × 7 = 28)
  – Odd × Odd = Odd (5 × 7 = 35)
- Assume √2 is Rational: $\sqrt{2} = {}^{P}/_{Q}$
- Assume Lowest Form: P, Q aren't <u>both</u> even
  – (if both were even, we can repeatedly divide both P and Q by 2 until at least one is odd)

Aside: Proof that √2 is Irrational

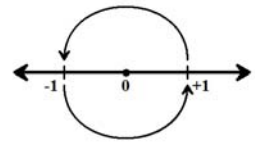
- Square both sides: $2 = P^2 / Q^2$
- Multiply by Q^2: $2Q^2 = P^2$
- Conclusion #1: P^2 is even, thus P is even
- Divide by 2: $Q^2 = P^2 / 2 = P \times {}^{P}/_{2}$
- Conclusion #2: Q^2 is even, thus Q is even
- Contradiction:
  – Initial assertion was P, Q <u>aren't both</u> even, proof says <u>both are</u> even, thus assumption that $\sqrt{2} = {}^{P}/_{Q}$ is false. No such rational number exists.

The Biggest Dirty Secret of Computing

- Most of the interesting numbers in the Universe are irrational,
- Numbers on computers have a fixed and finite number of bits,
- Therefore, most values get rounded off.
- **Most numerical results are approximations.**
- More bits means more precision, but only forestalls and does not eliminate the problem.

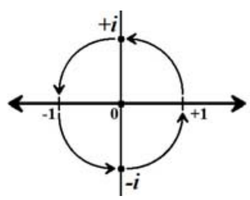
Complex Numbers

- The Real number line extends from -∞ to +∞,
- Use of space above and below the line gives us more computational expressive power.
- Negation then becomes a rotation of 180°:



Complex Numbers

- Rotation of +1 by 90° leaves it in space above the zero center. Call that number *i*:



Complex Numbers

- Multiplying a number by *i* twice equals negation,
- Thus $i^2 = -1$, and then $i = \sqrt{-1}$
- *i* is called “imaginary”

Complex Numbers

- A complex number is then a pair of numbers:
  - A value along the Real axis,
  - A value along the Imaginary axis.
  - Written with the Real part first, then Imaginary.
- Examples:
  - $2+3i$, $5-7i$, $-3+2i$, $-4-6i$, $6.7+5.9i$, etc.
  - 7 (same as $7+0i$)

Complex Math

- Add/Subtract: treat components separately:
  - $2+6i + 5-2i = (2+5) + (6-2)i = 7+4i$
  - $2+6i - 5-2i = (2-5) + (6--2)i = -3+8i$
- Multiplication uses FOIL method:
  - $2+6i \times 5-2i =$
  - $(2 \times 5) + (2 \times -2i) + (6i \times 5) + (6i \times -2i) =$
  - $(10) + (-4i) + (30i) + (-12i^2) =$
  - $(10+12) + (-4+30)i = 22+26i$

Complex Math

- Division uses <u>two</u> complex multiplications to eliminate imaginary component in denominator,
- Multiply both numerator and denominator by <u>complex conjugate</u> of denominator.
- Example:
  - $22+26i \div 2+6i =$
  - Numerator: $22+26i \times 2-6i = 200 - 80i$
  - Denominator: $2+6i \times 2-6i = 4+36 = 40$
  - $200 - 80i \div 40 = 5-2i$

Complex Math

- Used in math, engineering, physics, etc.
- Supported by early language FORTRAN,
- Supported by modern language Python,
- Supported (badly) by Excel 2007 and later.
- Mostly Double-Precision Floats,
- Subject to same round-off errors as other floating-point numbers.