Cops and Robbers with Traps and Doors

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July 20, 2012

General Rules

Cops and Robber is a game played on *reflexive graphs* that is, vertices each have a least one loop (which is why you can pass). There are two players: one consisting of the set of *cops* and a single *robber*.

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If after some finite number of rounds one of the cops is able to occupy the same vertex as the robber, the cops win. The robber wins if he can evade capture indefinitely. The cop number of a graph is the minimum number of cops to ensure that the cops win on that graph.

Traps and Doors

We are interested in the case when there are fewer cops than the cop number of the graph. The cops cannot catch the robber, so we give the cops some devices to use. The first kind is *traps* - if the robber enters a vertex with a trap, then the cops win. Traps can be picked up by cops and moved around the graph. The second kind is *doors* - these must be permanently placed before the game starts and the robber just can't enter vertices with them.

Traps and Doors

We use the following notation:
$\mathrm{tr}_n(G) = m$ means that the minimal number of traps required to win on G with n cops is m.
Similarly $\mathrm{dr}_n(G) = m$ stands for the number of doors.
$\mathrm{tr}(G)$ and $\mathrm{dr}(G)$ mean $\mathrm{tr}_1(G)$ and $\mathrm{dr}_1(G)$ respectively. There is an alternative notation for traps: G is an (n, m) win iff n cops can win with m traps. However, in this case m is not necessarily minimal.

Corners

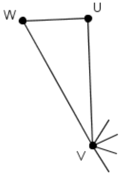
A vertex v is a corner iff there exists a vertex that is adjacent to v and every neighbor of v. Deleting a corner does not change the cop number of a graph. A graph is cop-win iff it can be reduced to a point by deleting corners. Graphs which can be reduced to a point by deleting corners are called *dismantleable* graphs.

Theorem

If G is a graph of girth at least 4 (triangle-free graph), then $\mathrm{dr}(G) = m$ iff deleting m vertices from G results in a tree or a forest.

Proof.

Forward direction:



Each corner in G has degree 1. The only dismantleable graph where all corners are of degree 1 is a tree. So if we have $\mathrm{dr}(G) \leq m$, then the robber has to be restricted to a tree. And then deleting vertices with doors would result in a tree (if H has one component) or a forest (since every component of H has to be a tree).

Proof.

Backward direction:

If we place doors on the vertices we would delete, then the robber would be restricted to a tree where the cop can catch him. □

This theorem shows that the problem of doors on triangle-free graphs is equivalent to the well-known feedback vertex set problem. A feedback vertex set of a graph is a set of vertices whose removal leaves a graph without cycles. For graphs with girth 3 the size of feedback vertex set would give an upper bound for the number of doors, but not necessarily the minimal number.

Theorem

If a set of vertices T can be deleted from a graph G so that for each connected component H of $G \setminus T$ $\exists G'$ such that $H \subseteq G' \subseteq G$ and G' is cop-win, then $\mathrm{dr}(G) \leq |T|$.

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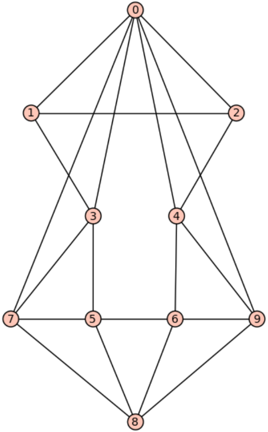
Proof.

Place a door on each vertex in T. The robber cannot pass through a vertex with a door, so the robber is restricted to moving on one connected component H. By assumption, there is a cop-win G' such that $H \subseteq G'$. If the cop plays his winning strategy on G' and the robber is restricted to moving on H, then the cop will catch the robber. Since the cop can catch the robber wherever he is after $|T|$ doors have been placed, $\mathrm{dr}(G) \leq |T|$. □

Note that the converse is false:



Conjecture

If G is a corner-free graph, then $\mathrm{dr}(G) = 1$ iff a vertex can be deleted from G to give a dismantleable graph.

Conjecture

If $\mathrm{dr}(G) = m$, then after m repetitions of the following algorithm starting with G, you will be left with a disjoint union of cop-win graphs.

1. Fully corner retract the graph
2. Delete a vertex

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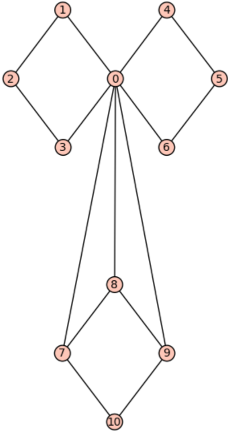

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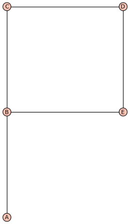

Proof.

A graph consisting of $n + 1$ copies of this graph with vertex A_i adjacent to vertex A_{i+1} for $i \leq n$ has $\mathrm{tr}(G) = 1$ and $\mathrm{dr}(G) = n + 1$. □

Definitions

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A *maximal planar graph* is a planar graph such that every face is bounded by exactly three edges, including the outer face. Maximal planar graphs are also referred to as *triangulated graphs*.

Maximal Planar Results

Theorem

For G a maximal planar graph, $|V(G)| \leq 11$, $\mathrm{dr}(G) \leq 1$.[a]

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For G a maximal planar graph, $|V(G)| \leq 11$, one cop can win on G if a door is placed on a maximum degree vertex.

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Theorem

For G a maximal planar graph, $|V(G)| \leq 11$, one cop can win on G if a door is placed on a maximum degree vertex.

Strategy: After placing a door on a maximum degree vertex, the resultant subgraph can be reduced to a point through a series of corner retractions.

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Maximal Planar Results

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For G a maximal planar graph, $12 \leq |\mathrm{V}(G)| \leq 13$, $\mathrm{dr}(G) \leq 2$.

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Note: Of the 125 $|V(G)| = 12$, corner-free maximal planar graphs, 3 have $dr(G) = 2$. Of the 494 $|V(G)| = 13$, corner-free maximal planar graphs, 8 have $dr(G) = 2$.

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For G a maximal planar graph, $12 \leq |V(G)| \leq 13$, $dr(G) \leq 2$.

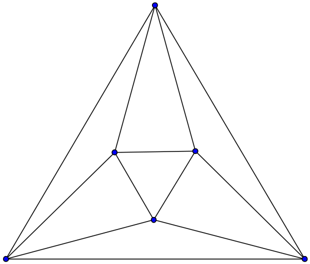
Note: Of the 125 $|V(G)| = 12$, corner-free maximal planar graphs, 3 have $dr(G) = 2$. Of the 494 $|V(G)| = 13$, corner-free maximal planar graphs, 8 have $dr(G) = 2$.

Strategy: Placing a door on a maximal degree vertex and then placing a door on an adjacent vertex will result in a two-door win.

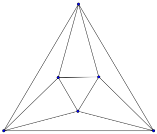
Octahedron

Theorem

The door number of the octahedron is 1.



Octahedron



Proof.

Let the cop place a door on any vertex of the octahedron. All vertices of the octahedron have degree 4. Deleting one vertex deletes 4 edges. Therefore one vertex is removed, four vertices are left with degree 3, and one vertex remains with degree 4, as this vertex was not adjacent to the removed vertex. The vertex with degree 4 is therefore a universal vertex. The cop can sit on the universal vertex and win, therefore making the door number of the octahedron 1. □

General Result

Theorem

The door number of $2n+1$ face connected octahedra is $n+1$.

Proof Sketch:

- The door number of the octahedron is 1.
- Placing a door on a shared vertex between two face connected octahedra will also allow the cop to win.
- n doors can dismantle $2n$ octahedra.
- Adding another octahedron can be won with the addition of one door, therefore the door number of $2n+1$ face connected octahedra is less than or equal to $n+1$.
- n doors are not sufficient to win on $2n+1$ face connected octahedra.
- It follows then that for all $n \geq 1$, there exists a maximal planar graph that is a $(1, n)$–win.

Open Questions

Can any results from maximal planar work be extended?

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Can it be proven formally that placing a door on a maximum degree vertex is a sufficient condition for a cop to win with one door for all maximal planar graphs $|V(G)| \leq 11$?

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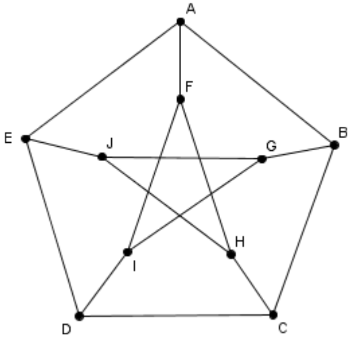
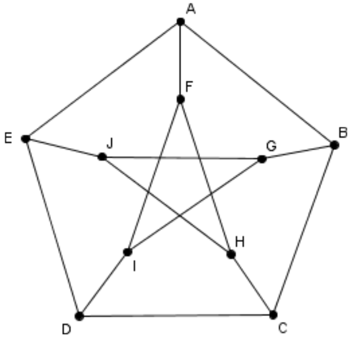
What is the smallest maximal planar graph with door number 3?

Petersen Graph



Petersen Graph



It is known that 3 cops are required to win on Petersen graph with no devices. We show that $\mathrm{tr}_2(P) = \mathrm{dr}_2(P) = 1$ and $\mathrm{tr}_1(P) = \mathrm{dr}_1(P) = 3$.

Theorem

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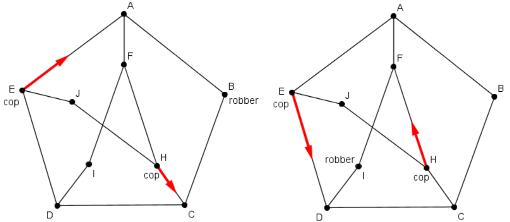
Proof.

Place a trap/door on vertex G and let the cops move to vertices E and H. The robber can only be on vertex B or I because other vertices are either occupied or adjacent to the cops. If the robber is on B, the cops move to A and C and the robber is now trapped between them. If the robber is on I, the cops move to D and F and the robber is trapped again. □



Theorem

$\mathrm{tr}_1(P) = \mathrm{dr}_1(P) = 3$

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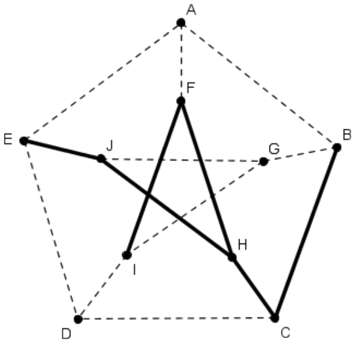
Proof.

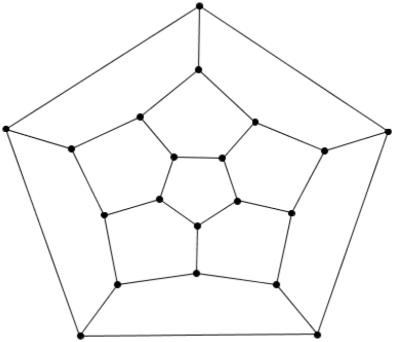
Place traps/doors on vertices A, D, and G. Now the vertices left for the robber form a tree and therefore the cop can capture him.



Proof.

Now we show that 3 is the minimal number of traps/doors. Assume the cop has only 2 traps. All vertices of Petersen graph have degree 3. To trap the robber at some vertex v, the cop has to place traps on 2 neighbors of v and then go around and occupy the third neighbor. Therefore, we don't have to move traps and they are equivalent to doors. However, 2 doors can't eliminate all the 5-cycles and the vertices where the robber can move still have a cycle. Therefore, the cop can't capture the robber. □

Dodecahedron



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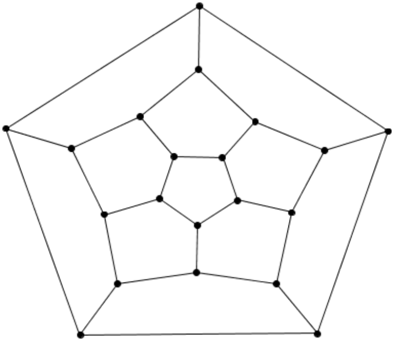
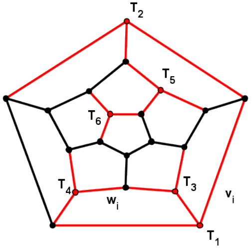


It is known that 3 cops are required to win on the dodecahedron graph with no devices. We show that $\mathrm{tr}_2(D) = \mathrm{dr}_2(D) = 1$ and $\mathrm{tr}_1(D) \leq \mathrm{dr}_1(D) = 6$.

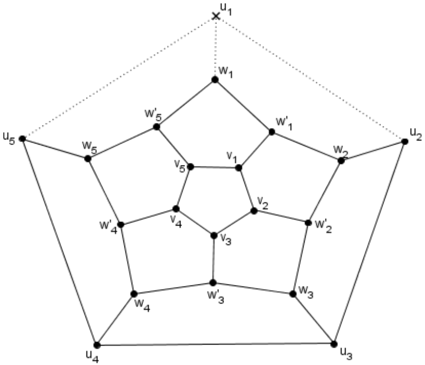
Theorem

$\mathrm{tr}_1(D) \leq \mathrm{dr}_1(D) = 6$



Theorem

$\mathrm{tr}_2(D) = \mathrm{dr}_2(D) = 1$



We would like to thank:

- Central Michigan University and Dr. Sivaram Narayan for organizing this program.
- The NSF grant DMS 11-56890 for funding this program
- Dr. Erin Militzer and Dr. Jordan Webster for mentoring us.

Questions?