Solving Graph Problems in the Streaming Model

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Algorithms for Massive Graphs

Graph algorithms have a variety of applications across many areas of computer science and other fields.

Problem: the graph in question can be

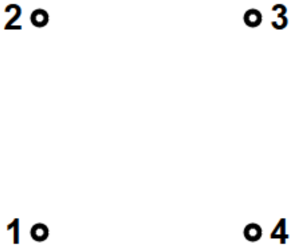
- too large to be stored in main memory
- distributed across many machines
- changing over time

Streaming to the rescue!

Streaming Model: Insert-Only

Edges of the graph arrive one-by-one in a sequence.

In an **insert-only** stream we only insert edges into the graph.



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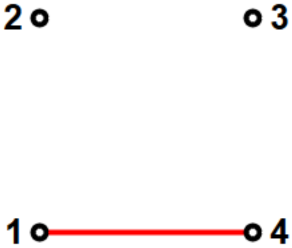

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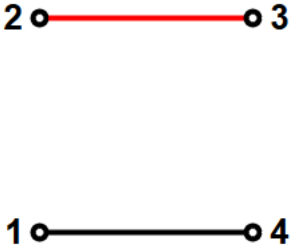

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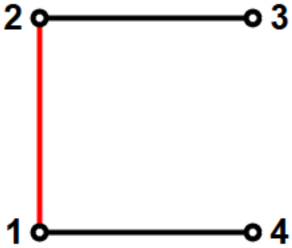

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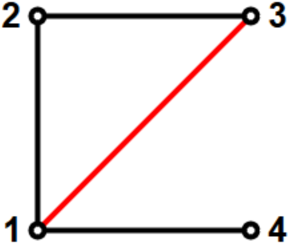

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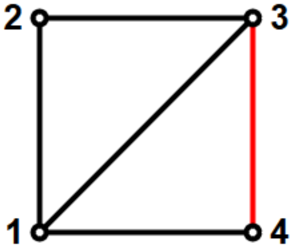

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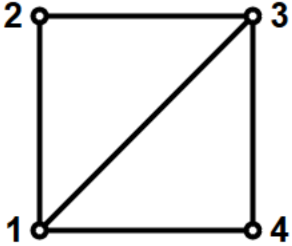
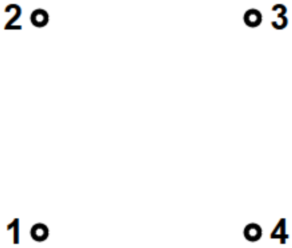


$(1,4)$, $(2,3)$, $(1,2)$, $(1,3)$, $(3,4)$

Streaming Model: Dynamic

Edges of the graph arrive one-by-one in a sequence.

In an **insert-delete** or **dynamic** stream we can both insert and delete edges.



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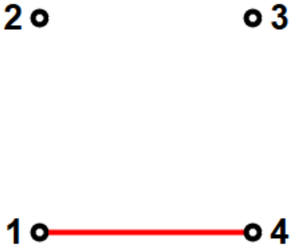

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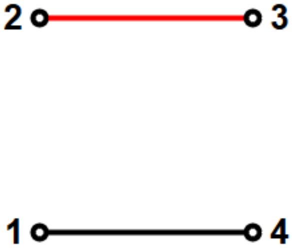

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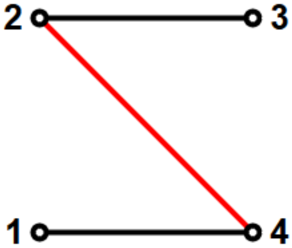

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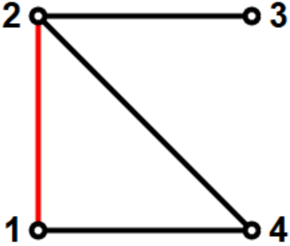

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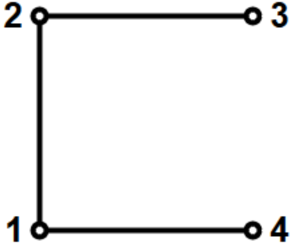

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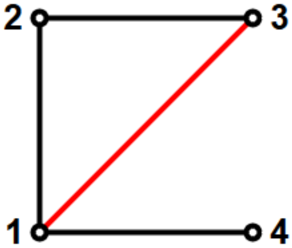

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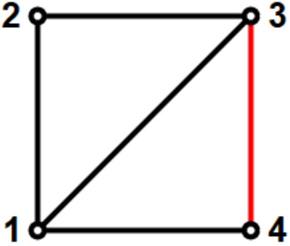

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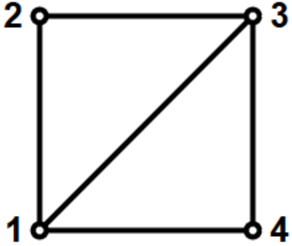

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Streaming Model: Adjacency List

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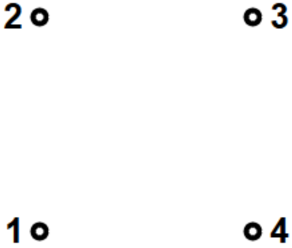

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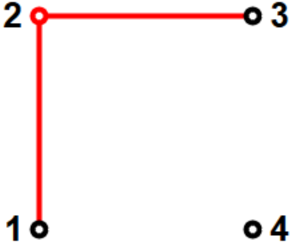

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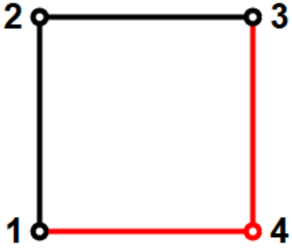

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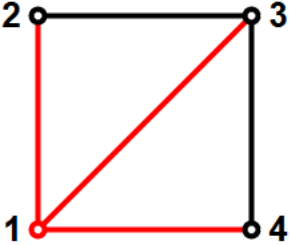

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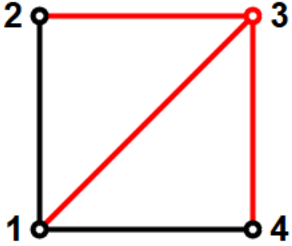

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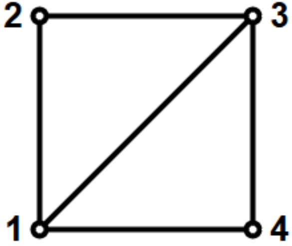

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Graph Problems in the Streaming Model

Objective: minimize the amount of information we store throughout the stream that is still sufficient to (approximately) solve the problem.

- Densest subgraph
- Vertex cover
  - Solution size is small
- Maximum matching
  - Solution size is small
  - Input graph is sparse
- Counting triangles

Densest Subgraph

Density of a graph is defined as $\frac{\#\text{edges}}{\#\text{vertices}}$. Given an input graph, want to find its subgraph with maximum density.

Used for community detection in social networks and identifying link spam on the web, in addition to applications on financial and biological data.

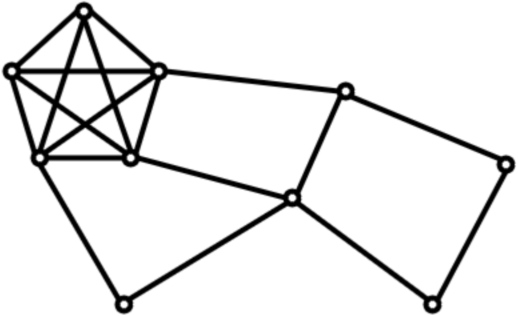
If we have unlimited access to the entire graph, this problem can be solved in polynomial time.

We show[1] that for an arbitrary order stream (insertion-only or dynamic) we can find a $(1+\epsilon)$-approximation while storing $\tilde{O}(n)$ edges.

Densest Subgraph: Main Idea

Sample $\tilde{O}(n)$ edges uniformly at random.

- If a subgraph has small density, after sampling its density remains relatively small
- If a subgraph has high density, after sampling its density remains relatively high



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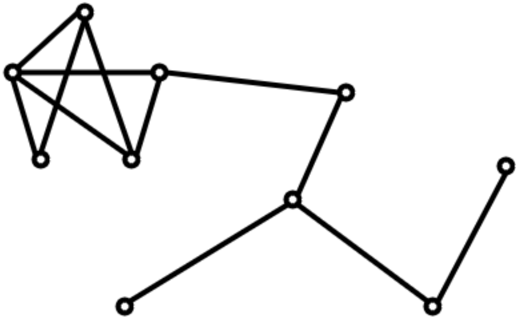


Vertex Cover and Matching of Small Size

Vertex cover of a graph is a set of vertices s.t. every edge has at least one endpoint in the set.
Matching is a set of edges that don't share endpoints.

We want to find a vertex cover of minimum size and a matching of maximum size.

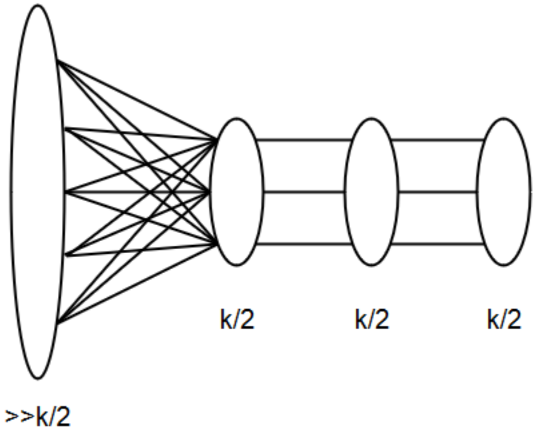
The problems are closely related:

$$\mathrm{match}(G) \leq \mathrm{vc}(G) \leq 2\,\mathrm{match}(G)$$

We show[2] that if the size of min vertex cover/max matching is at most k, then we can find it exactly using $\tilde{O}(k^2)$ space in insert-only or dynamic arbitrary order stream.

Vertex Cover and Matching of Small Size

Uniform sampling doesn't work in this case.



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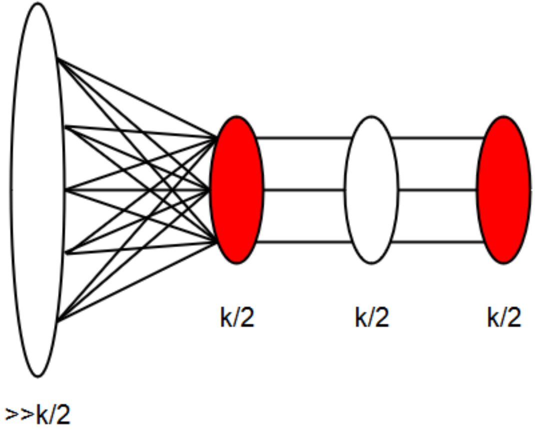

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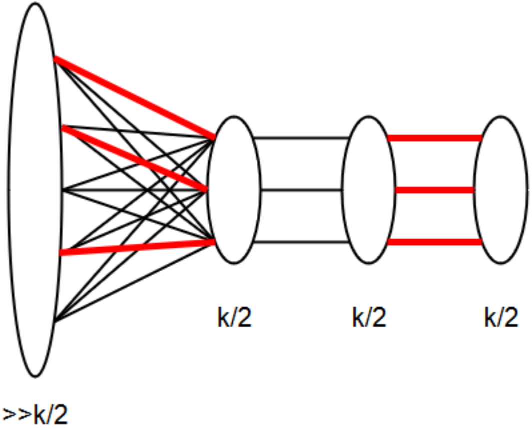

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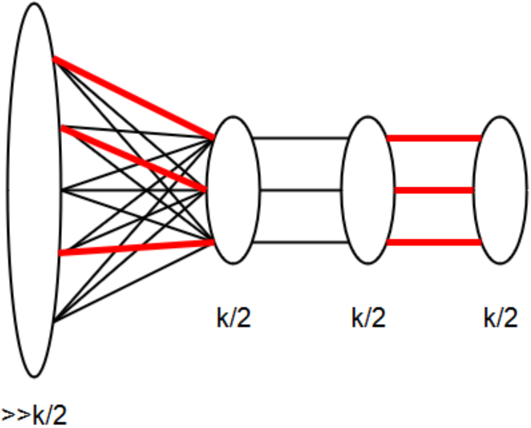
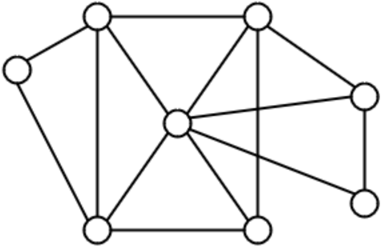


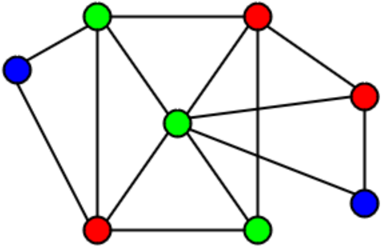
Problem: Oversampling from dense parts of the graph. This was good for densest subgraph, but not here.

Vertex Cover and Matching of Small Size: Color Sampling



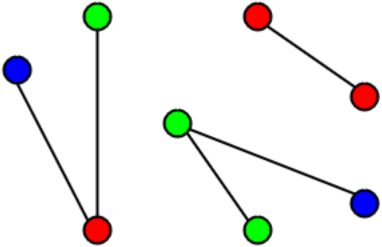
Vertex Cover and Matching of Small Size: Color Sampling

- Color vertices with b colors.



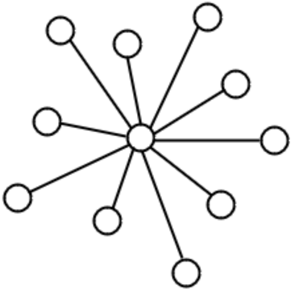
Vertex Cover and Matching of Small Size: Color Sampling

- Color vertices with b colors.
- For every color pair, sample one edge uniformly at random.



Vertex Cover and Matching of Small Size: Color Sampling

This avoids oversampling from high degree vertices.



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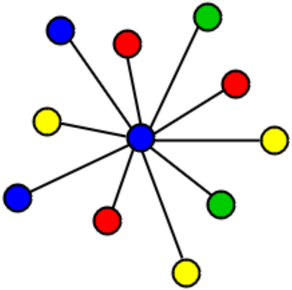

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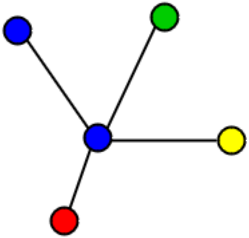
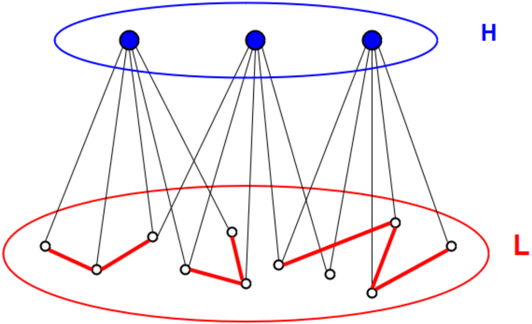


Vertex Cover and Matching of Small Size

Assume $\text{match}(G) \leq \text{vc}(G) \leq k$

H = set of **heavy** vertices with degree at least $10k$

L = set of **light** edges s.t. both endpoints are not heavy



$$\text{vc}(G) = |H| + \text{vc}(L)$$

$$\text{match}(G) = |H| + \text{match}(L)$$

Vertex Cover and Matching of Small Size

Let G' be the graph obtained by sampling using $\Theta(k)$ colors.

With constant probability:

- If an edge is light, then it is sampled
- If a vertex is heavy, its degree in G' is at least $5k$, which is high enough to recognize that it is heavy

By repeating the sampling $\Theta(\log n)$ times we sample all light edges and detect all heavy vertices with high probability.

Thus, preserving the size of vertex cover and matching.

Approximating Size of Maximum Matching

In the previous result we were finding the matching itself. However, if there are no size restrictions, maximum matching can be as large as $n/2$.

By approximating the **size** of the matching without finding the matching itself, we can use smaller space.

Approximating Size of Maximum Matching: Arboricity

We concentrate on the class of graphs of arboricity α.

Arboricity is the minimum number of forests into which the edges of the graph can be partitioned.
Property: In a graph with arboricity α, every induced subgraph on r vertices has at most αr edges.

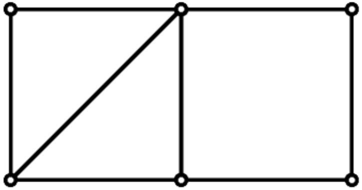
Planar graphs have arboricity at most 3.

We show[3] an $(\alpha + 2)$-approximation using space

- $\tilde{O}(\alpha n^{2/3})$ in insert-only arbitrary order streams
- $\tilde{O}(\alpha n^{4/5})$ in dynamic arbitrary order streams

Approximating Size of Maximum Matching: Local Fractional Matching

$$x_{uv} = \min\left(\frac{1}{deg(u)}, \frac{1}{deg(v)}, \frac{1}{\alpha+1}\right)$$



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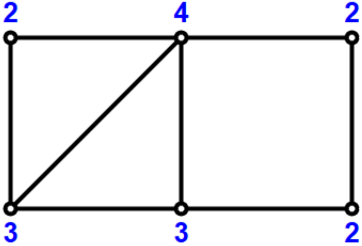

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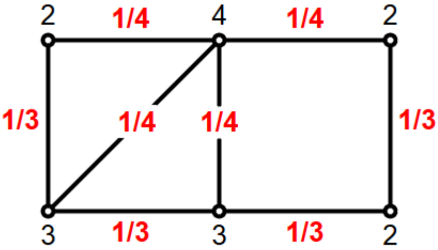

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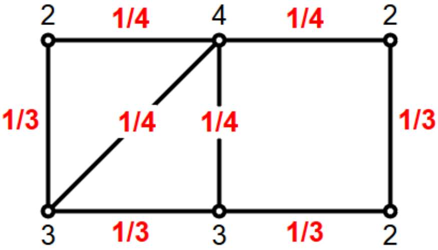


- $\mathbf{x}$ is a fractional matching
- x_e is only a function of edges that share an endpoint with e
- $(\alpha+1)\sum_e x_e$ is an $(\alpha+2)$-approx of match(G)

Approximating Size of Maximum Matching: Implementation in Graph Streams

Sample a set of vertices S uniformly at random with probability p. Let E_S be edges with both endpoints in S. In parallel:

- Compute (approximate) matching if it is small
- If the matching is large, use $\frac{1}{p^2}\sum_{e\in E_S} x_e$ as estimate

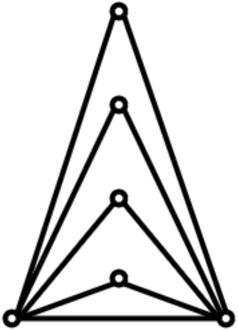
We balance the size of set S and space required to find (approximate) small matching to get space

- $\tilde{O}(\alpha n^{2/3})$ in insert-only streams
- $\tilde{O}(\alpha n^{4/5})$ in dynamic streams

Triangle Counting

Problem: count/approximate the number of triangles in a graph.

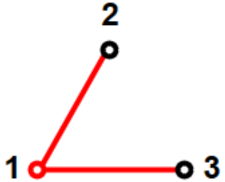
Hard in arbitrary order streaming!



THE TOWER OF DOOM

Triangle Counting in Adjacency List Streams

Counting triangles in **adjacency list streams is easier**, since we see every triangle twice.



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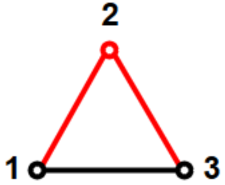

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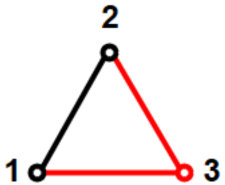


Triangle Counting: Results

We parametrize the algorithm in terms of T, the number of triangles. This convention is widely adopted in the literature. The problem can then be viewed as distinguishing between graphs with at most t and at least $(1+\epsilon)t$ triangles.

In the insert-only adjacency list stream we obtain[4] space

- $\tilde{O}(\frac{m}{\sqrt{T}})$ in one pass
- $\tilde{O}(\frac{m^{3/2}}{T})$ in two passes

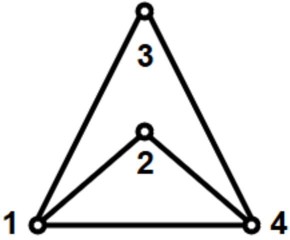
Note that

$$\frac{m}{\sqrt{T}} \leq \frac{m^{3/2}}{T} \text{ when } T \leq m \text{ (small number of triangles)}$$

$$\frac{m}{\sqrt{T}} \geq \frac{m^{3/2}}{T} \text{ when } T \geq m \text{ (large number of triangles)}$$

Triangle Counting: One Pass Algorithm

Assign each triangle to one of the edges. A triangle xyz where the vertices appear in order x, then y, then z in the stream, is assigned to edge xz. Call the number of triangles assigned to xz, R_{xz}.

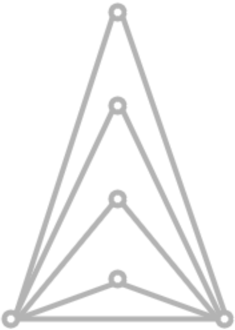


$R_{1,4} = 2$

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Large tower



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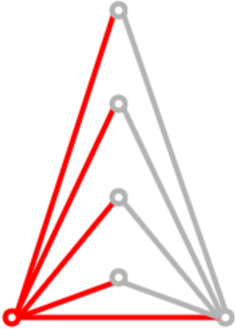
Large tower

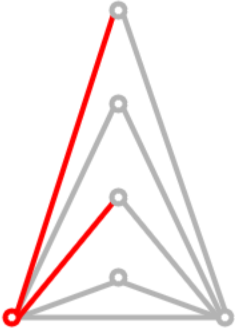


When the first vertex of the tower arrives, sample edges uniformly.

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Large tower



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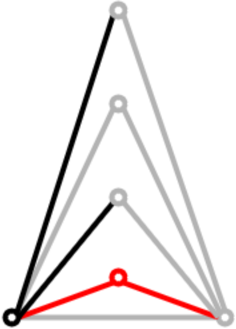
Main idea: estimate large R_{xz} separately from small R_{xz}.

Large tower

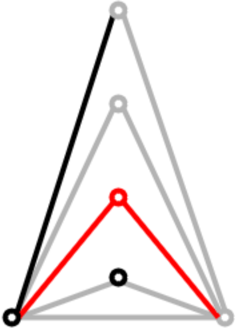


For middle vertices, do nothing.

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Large tower

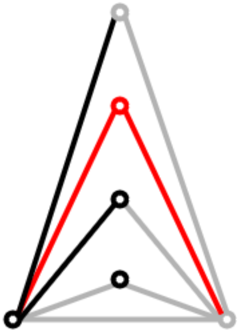


For middle vertices, do nothing.

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Large tower



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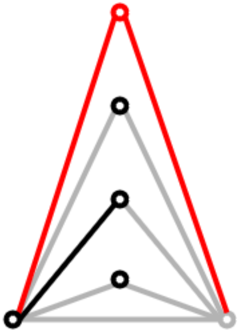
Main idea: estimate large R_{xz} separately from small R_{xz}.

Large tower



For middle vertices, do nothing.

Triangle Counting: One Pass Algorithm

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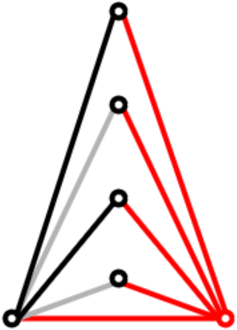
Large tower

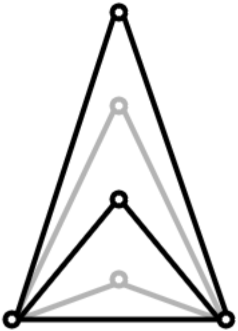


For the last vertex, count the number of triangles formed.

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Large tower

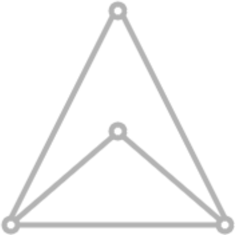


For the last vertex, count the number of triangles formed.
Since the tower is large, that gives an accurate estimate.

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

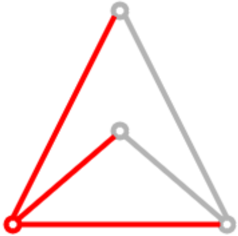
Small tower



Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Small tower

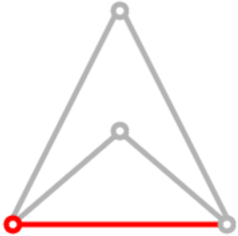


When the first vertex of the tower arrives, sample edges uniformly (only care about the base of the tower).

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Small tower



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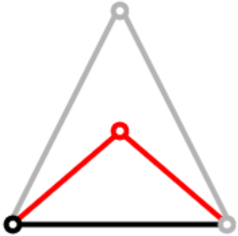
Small tower

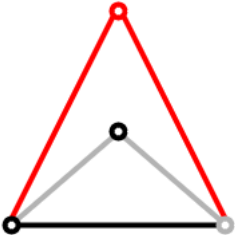


If the base was sampled: for middle vertices, increment the counter.
$R_{xz} = 1$

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Small tower

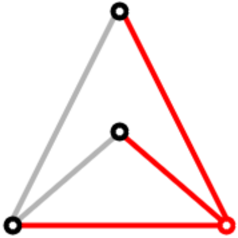


If the base was sampled: for middle vertices, increment the counter.
$R_{xz} = 2$

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

Small tower

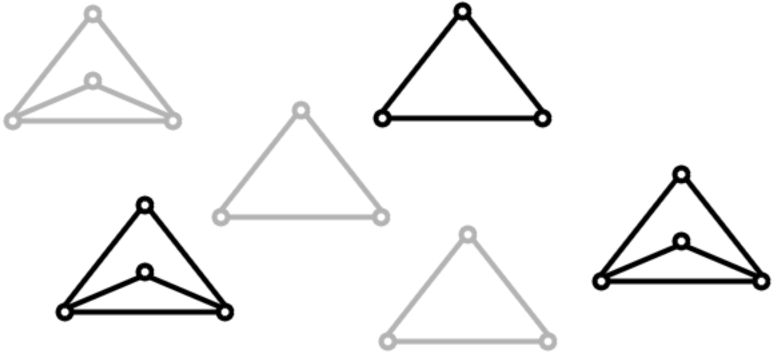


For the last vertex, do nothing.
$R_{xz} = 2$

Triangle Counting: One Pass Algorithm

Main idea: estimate large R_{xz} separately from small R_{xz}.

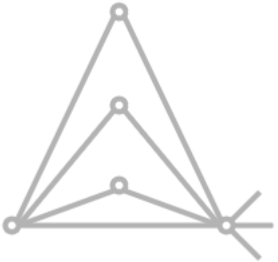
Small towers



Depend on whether the base was sampled.
Since these are small, the estimate is still accurate.

Triangle Counting: Two Pass Algorithm

Again, we assign each triangle to one of the edges. A triangle xyz with vertex degrees $\deg(x) \le \deg(y) \le \deg(z)$, is assigned to edge xy. Call the number of triangles assigned to xy, R_{xy}.



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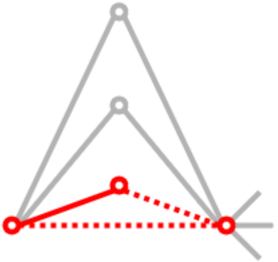

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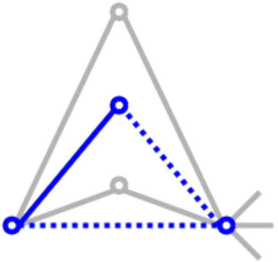

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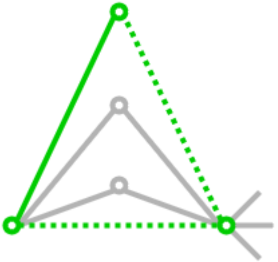
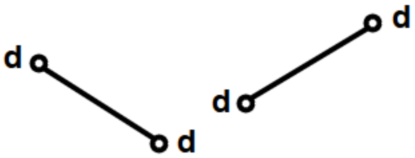


Triangle Counting: Two Pass Algorithm

Main idea: All R_{xy} are now relatively small. $R_{xy} \leq \sqrt{2m}$ for all xy.

First pass



- sample each edge uniformly at random
- store the sampled edges and degrees of their endpoints

Triangle Counting: Two Pass Algorithm

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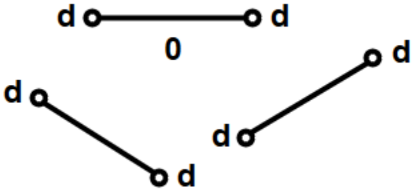
Second pass



- Set counters for all sampled edges to 0

Triangle Counting: Two Pass Algorithm

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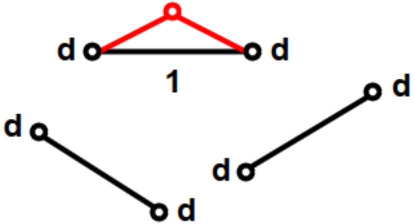
Second pass



- v arrives: check all sampled edges ab
- vab is Δ, $\deg(a) \leq \deg(v)$, $\deg(b) \leq \deg(v)$: increment R_{ab}

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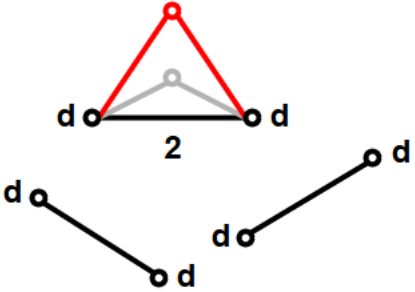
Second pass



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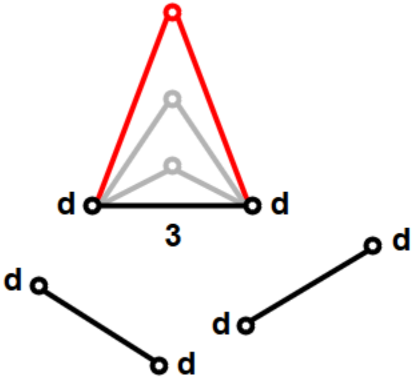

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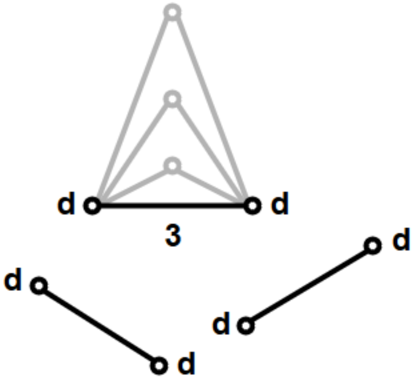

- Since all towers are small, the estimate is accurate.

Conclusion: Things To Do With Streams

Sampling!

- Sample edges uniformly.
- Sample edges non-uniformly. Example: color sampling.
- Sample vertices, then use the induced subgraph.

Other things:

- Compute degrees of vertices or other quantities depending on degrees.
- Using stream ordering as part of the algorithm.

Thank you for your attention!

Questions?

Bibliography

1 *Densest Subgraph in Dynamic Graph Streams*. A. McGregor, D. Tench, S. Vorotnikova, and H. Vu. MFCS 2015

2 *Kernelization via Sampling with Applications to Finding Matchings and Related Problems in Dynamic Graph Streams*. R. Chitnis, G. Cormode, H. Esfandiari, M. Hajiaghayi, A. McGregor, M. Monemizadeh, and S. Vorotnikova. SODA 2016

3 *Planar Matchings in Streams Revisited*. A. McGregor and S. Vorotnikova. APPROX 2016

4 *Better Algorithms for Counting Triangles in Data Streams*. A. McGregor, S. Vorotnikova, and H. Vu. PODS 2016