Decision Trees

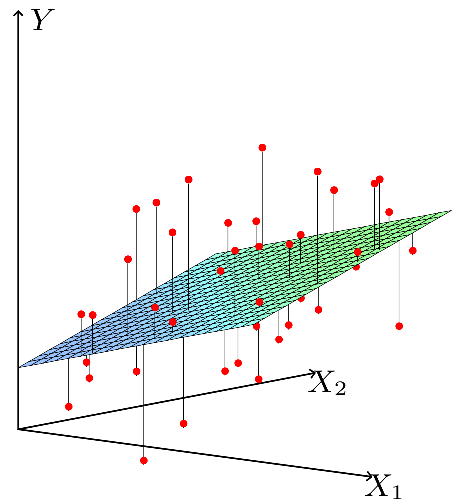
CS 341 – Lectures 8/9

Dan Sheldon

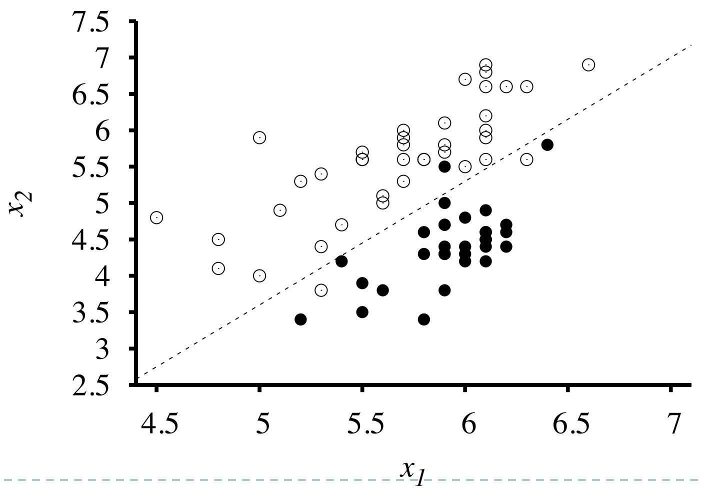
$h(x) = w_0 + w_1 x_1 + w_2 x_2 + \dots + w_d x_d$

Review: Linear Methods

- So far, we've looked at *linear* methods

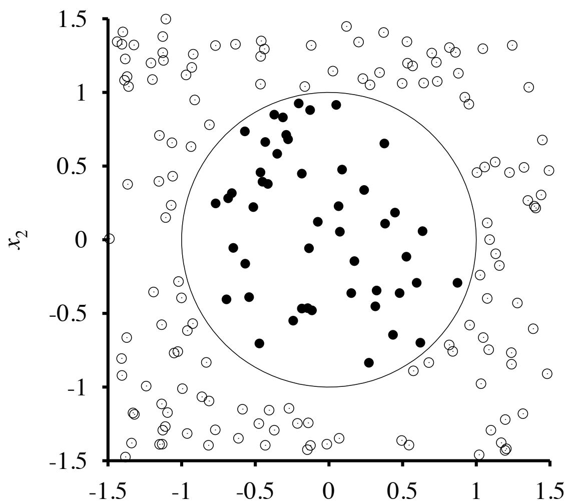


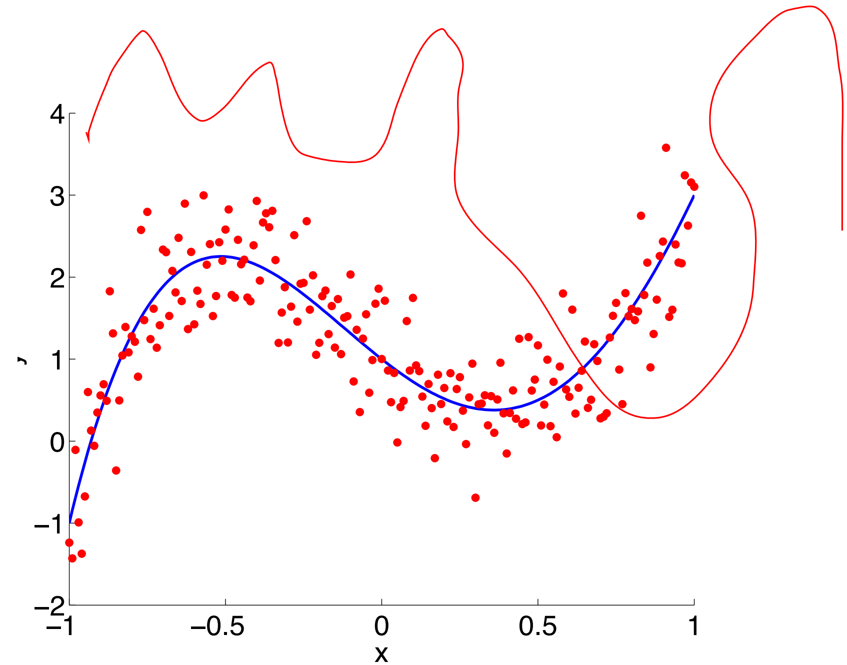
- Linear regression
  - Fit a line/plane/hyperplane
- Logistic regression
  - Decision boundary is a line/plane/hyperplane



Review: Feature Expansions

- Non-linearity by trickery
- Drawback: not automatic
  → hard work for us!





$$x \mapsto (1, x, x^2, x^3, \ldots)$$

sin(x)

$$(x_1, x_2) \mapsto (1, x_1, x_2, x_1^2, x_2^2, x_1 x_2)$$

$x_1^2 + x_2^2 = 1$

Next Up: Non-Linear Methods

- Hypothesis class is intrinsically non-linear

- Next few topics
  - Decision trees
  - Neural networks
  - Kernels

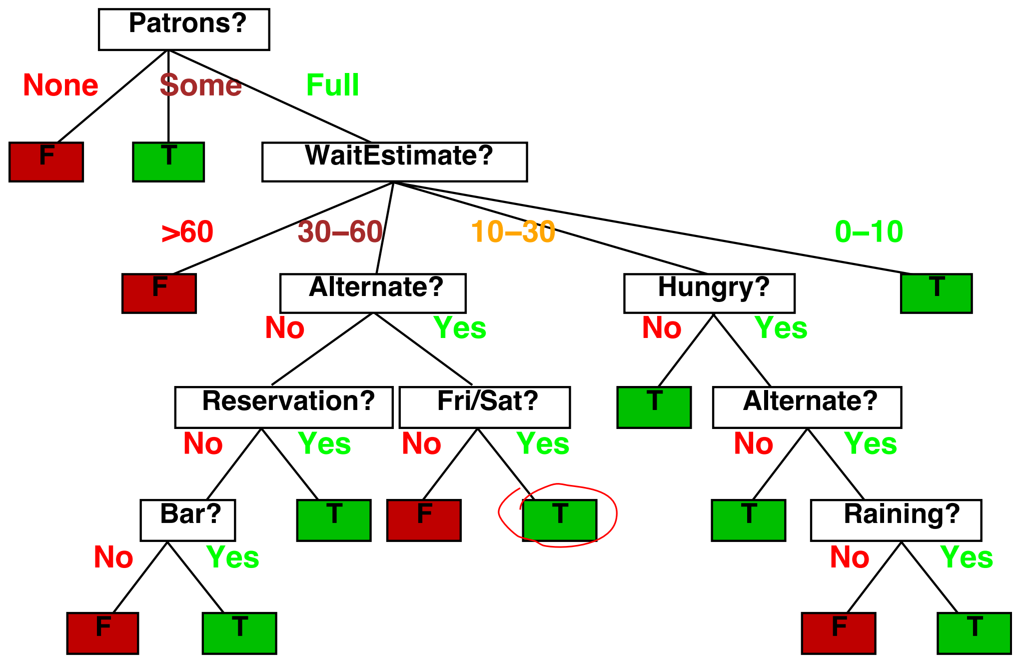
- Today: decision trees

Decision Tree Topics

- What is a decision tree?
- Learn a tree from data
  - Good vs. bad trees
  - Greedy top-down learning
  - Entropy
- Pruning
- Geometric interpretation

What Is a Decision Tree?

- Example from R&N: wait for table at restaurant?



[Figure: Russell & Norvig]

What Is a Decision Tree?

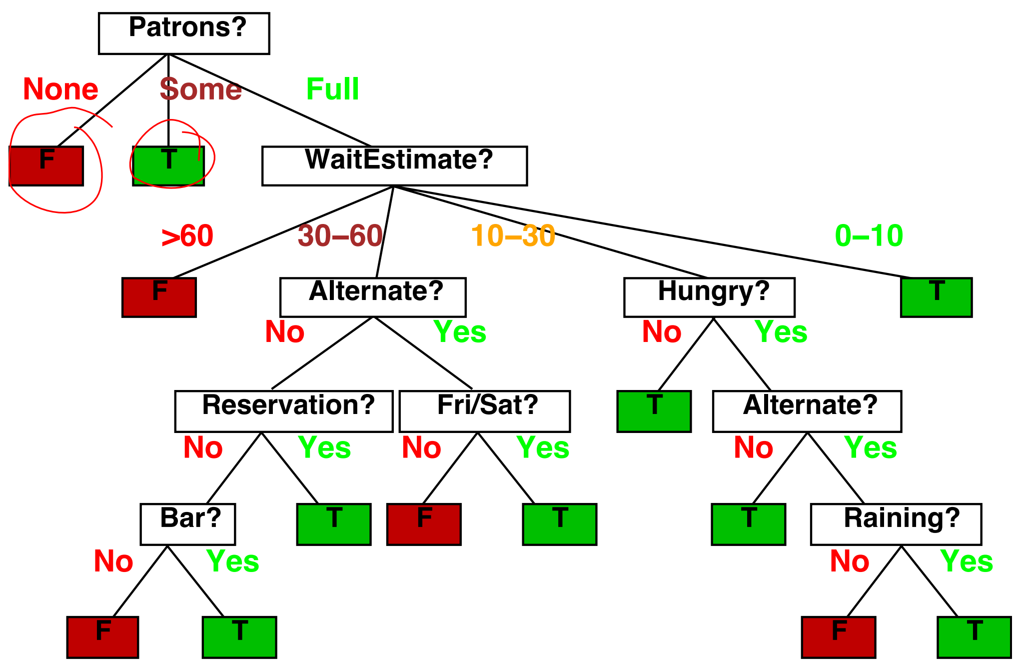
$\vec{x} \in \mathbb{R}^d$

$y \in \{0, 1\}$

- Internal nodes: test an attribute (feature)
  - Assume discrete attributes for now
- Branches: different values of attribute
- Leaf node: predict a class label

$\vec{x} = (x_1, x_2, \ldots, x_d)$

attributes

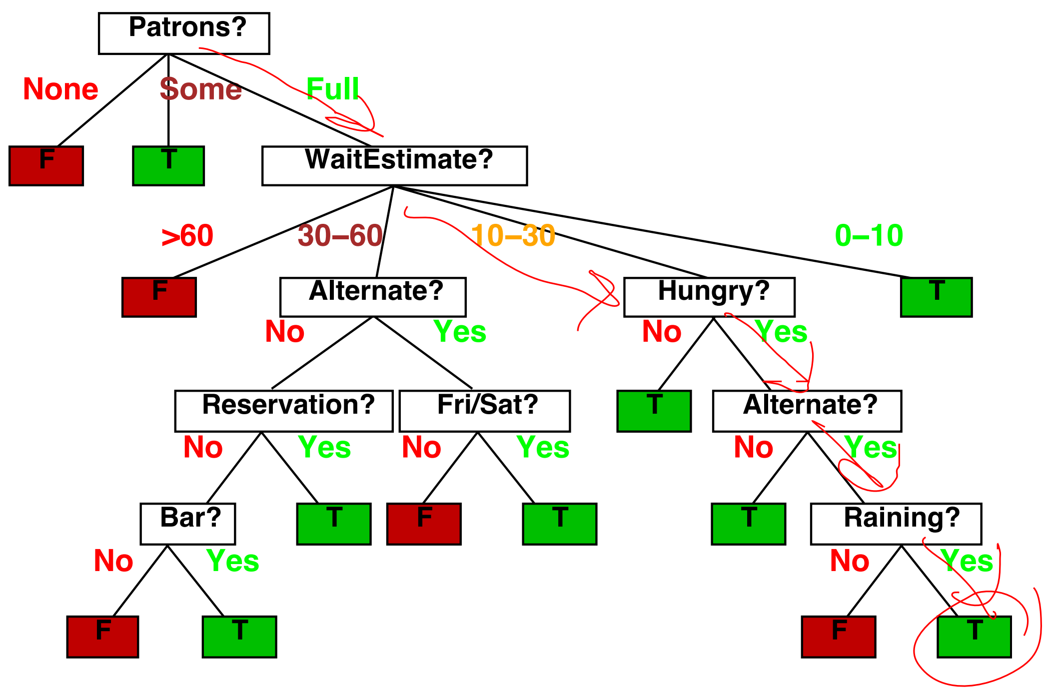


[Figure: Russell & Norvig]

What Is a Decision Tree?

- Predict by following path down tree

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*Wait?*
X_4	*T*	*F*	*T*	*T*	*Full*	*\$*	~~*F*~~ T	*F*	*Thai*	*10–30*	*T*



Decision Trees in the Wild

- Decision trees are common decision-making tools
  - You have all seen them!
- R&N: Car service manuals
- Viburnum ID
  - http://www.hort.cornell.edu/vlb/key/index.html

[Image: Clemson cooperative extension]

Decision Trees in the Wild

- C-section risk (learned from data)

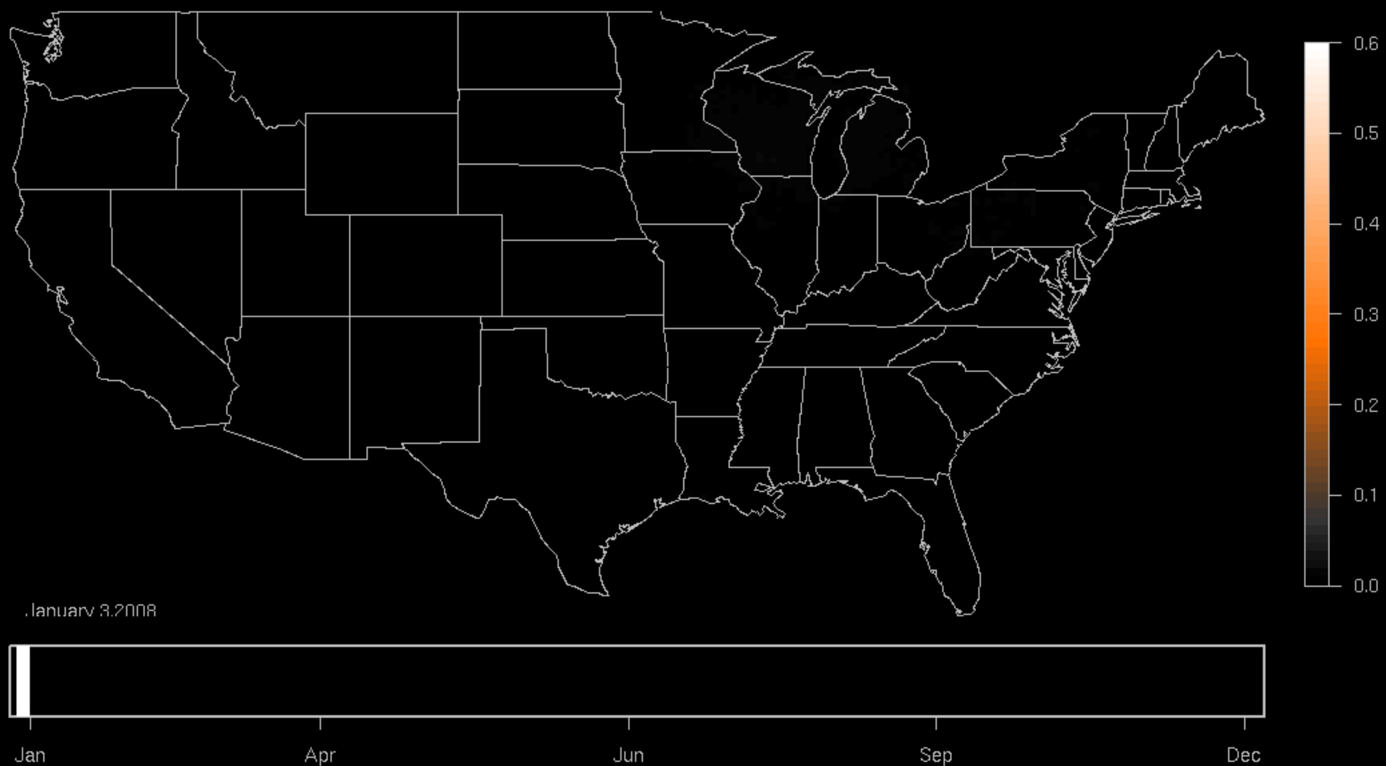
Learned from medical records of 1000 women

Negative examples are C-sections

```
[833+,167-] .83+ .17-
Fetal_Presentation = 1: [822+,116-] .88+ .12-
| Previous_Csection = 0: [767+,81-] .90+ .10-
| | Primiparous = 0: [399+,13-] .97+ .03-
| | Primiparous = 1: [368+,68-] .84+ .16-
| | | Fetal_Distress = 0: [334+,47-] .88+ .12-
| | | | Birth_Weight < 3349: [201+,10.6-] .95+ .
| | | | Birth_Weight >= 3349: [133+,36.4-] .78+
| | | Fetal_Distress = 1: [34+,21-] .62+ .38-
| Previous_Csection = 1: [55+,35-] .61+ .39-
Fetal_Presentation = 2: [3+,29-] .11+ .89-
Fetal_Presentation = 3: [8+,22-] .27+ .73-
```

[Example: X. Fern, Oregon State]

Decision Trees in the Wild



eBird

Why Decision Trees?

- Easy to understand
  - Match human decision-making processes
- Excellent performance
  - One of the best out-of-the-box methods when used carefully
- Very flexible hypothesis space
  - Can learn complex decision boundaries
- Handle messy data well
  - Missing features
  - Continuous discrete
  - Etc. No normalization

How to Learn a Tree From Data

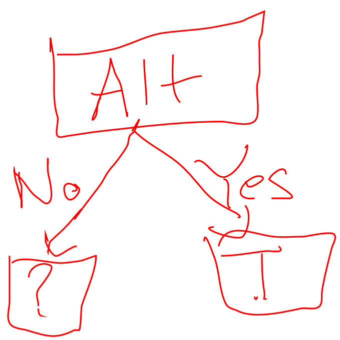
- Training data for R&N example

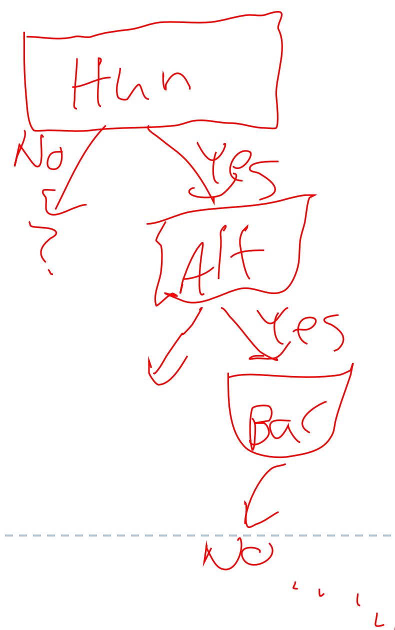
Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*Wait?*
X_1	*Yes*	*No*	*No*	*Yes*	*Some*	*$$$*	*No*	*Yes*	*French*	*0–10*	*T*
X_2	*Yes*	*No*	*No*	*Yes*	*Full*	*$*	*No*	*No*	*Thai*	*30–60*	*F*
X_3	*No*	*Yes*	*No*	*No*	*Some*	*$*	*No*	*No*	*Burger*	*0–10*	*T*
X_4	*Yes*	*No*	*Yes*	*Yes*	*Full*	*$*	*Yes*	*No*	*Thai*	*10–30*	*T*
X_5	*Yes*	*No*	*Yes*	*No*	*Full*	*$$$*	*No*	*Yes*	*French*	>60	*F*
X_6	*No*	*Yes*	*No*	*Yes*	*Some*	*$$*	*Yes*	*Yes*	*Italian*	*0–10*	*T*
X_7	*No*	*Yes*	*No*	*No*	*None*	*$*	*Yes*	*No*	*Burger*	*0–10*	*F*
X_8	*No*	*No*	*No*	*Yes*	*Some*	*$$*	*Yes*	*Yes*	*Thai*	*0–10*	*T*
X_9	*No*	*Yes*	*Yes*	*No*	*Full*	*$*	*Yes*	*No*	*Burger*	>60	*F*
X_{10}	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*$$$*	*No*	*Yes*	*Italian*	*10–30*	*F*
X_{11}	*No*	*No*	*No*	*No*	*None*	*$*	*No*	*No*	*Thai*	*0–10*	*F*
X_{12}	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*$*	*No*	*No*	*Burger*	*30–60*	*T*

Exercise

- Design an algorithm
  - Input: single training example
  - Output: decision tree that matches that example

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*Wait?*
X_4	*Yes*	*No*	*Yes*	*Yes*	*Full*	*\$*	*Yes*	*No*	*Thai*	*10–30*	*T*





Exercise (cont')

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*Wait?*
X_4	*Yes*	*No*	*Yes*	*Yes*	*Full*	*\$*	*Yes*	*No*	*Thai*	*10–30*	*T*

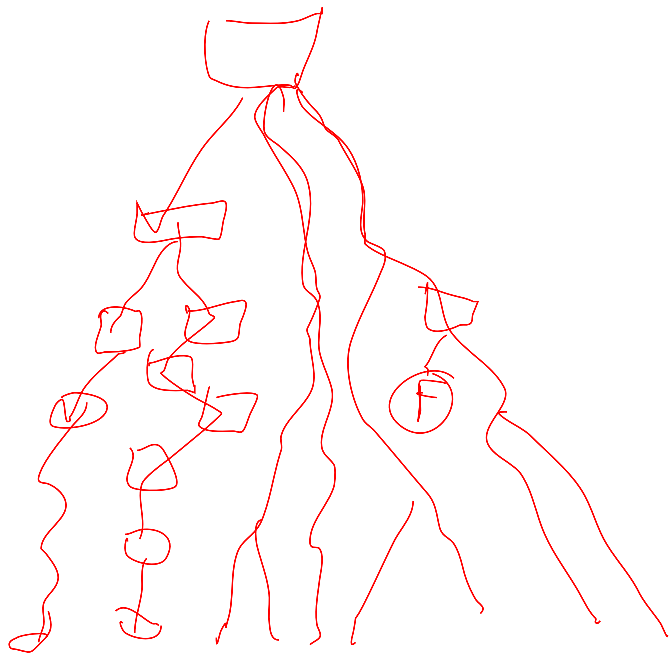
Exercise

- Design a tree to match *all* examples
  - Hint: use previous exercise

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*Wait?*
X_1	*Yes*	*No*	*No*	*Yes*	*Some*	*\$\$\$*	*No*	*Yes*	*French*	*0–10*	*T*
X_2	*Yes*	*No*	*No*	*Yes*	*Full*	*\$*	*No*	*No*	*Thai*	*30–60*	*F*
X_3	*No*	*Yes*	*No*	*No*	*Some*	*\$*	*No*	*No*	*Burger*	*0–10*	*T*
X_4	*Yes*	*No*	*Yes*	*Yes*	*Full*	*\$*	*Yes*	*No*	*Thai*	*10–30*	*T*
X_5	*Yes*	*No*	*Yes*	*No*	*Full*	*\$\$\$*	*No*	*Yes*	*French*	*>60*	*F*
X_6	*No*	*Yes*	*No*	*Yes*	*Some*	*\$\$*	*Yes*	*Yes*	*Italian*	*0–10*	*T*
X_7	*No*	*Yes*	*No*	*No*	*None*	*\$*	*Yes*	*No*	*Burger*	*0–10*	*F*
X_8	*No*	*No*	*No*	*Yes*	*Some*	*\$\$*	*Yes*	*Yes*	*Thai*	*0–10*	*T*
X_9	*No*	*Yes*	*Yes*	*No*	*Full*	*\$*	*Yes*	*No*	*Burger*	*>60*	*F*
X_{10}	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*\$\$\$*	*No*	*Yes*	*Italian*	*10–30*	*F*
X_{11}	*No*	*No*	*No*	*No*	*None*	*\$*	*No*	*No*	*Thai*	*0–10*	*F*
X_{12}	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*\$*	*No*	*No*	*Burger*	*30–60*	*T*

A Correct Answer

- For each training example, create a path from root to leaf
  - Use attributes in fixed order



leaves

→ training examples

→ or ? guesses

- What is wrong with this?

→ generalizes poorly

What Is a Good Tree?

- Find $\boxed{\textit{smallest}}$ tree that fits training data

  min # internal nodes
  leaves

- A very hard problem
  - Number of trees on d binary attributes *at least*

$$2^{2^d}$$

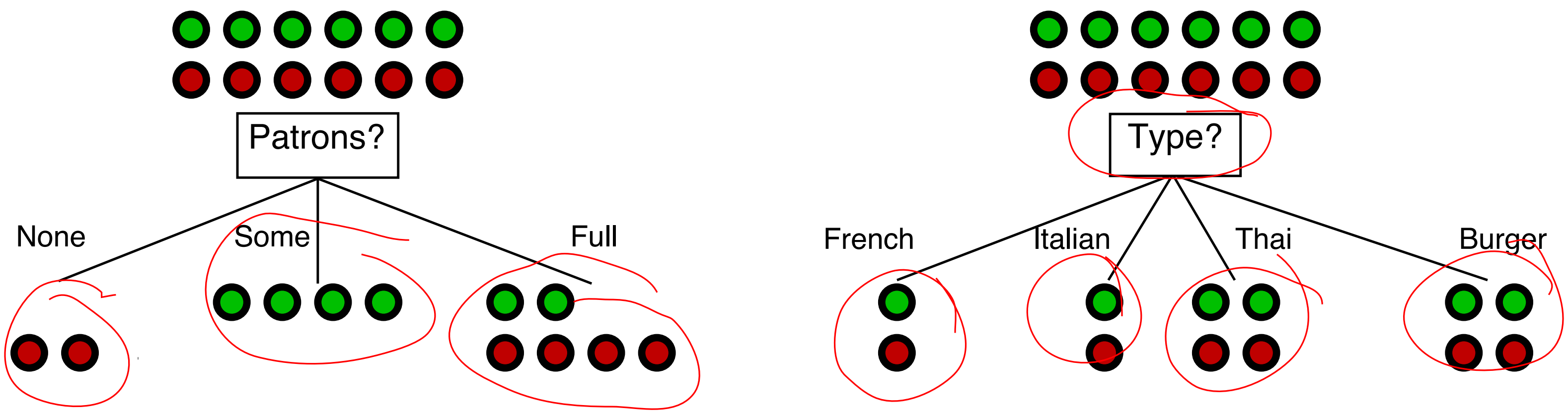
  - E.g., with 6 attributes

$$2^{2^6} = 18{,}446{,}744{,}073{,}709{,}551{,}616$$

- Effectively impossible to find *best* tree

Top-Down Greedy Heuristic

- Idea:
  - Find "best" attribute to test at root of tree
  - Recursively apply algorithm to each branch

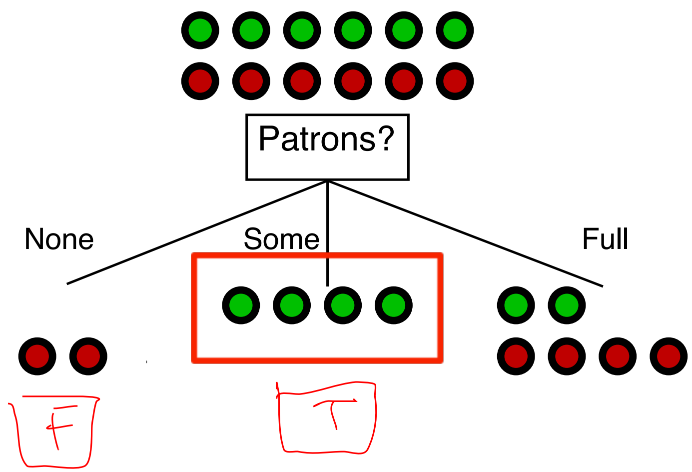


- "Best" attribute?
  - Ideally, splits the examples into subsets that are "all positive" or "all negative"

Base Cases

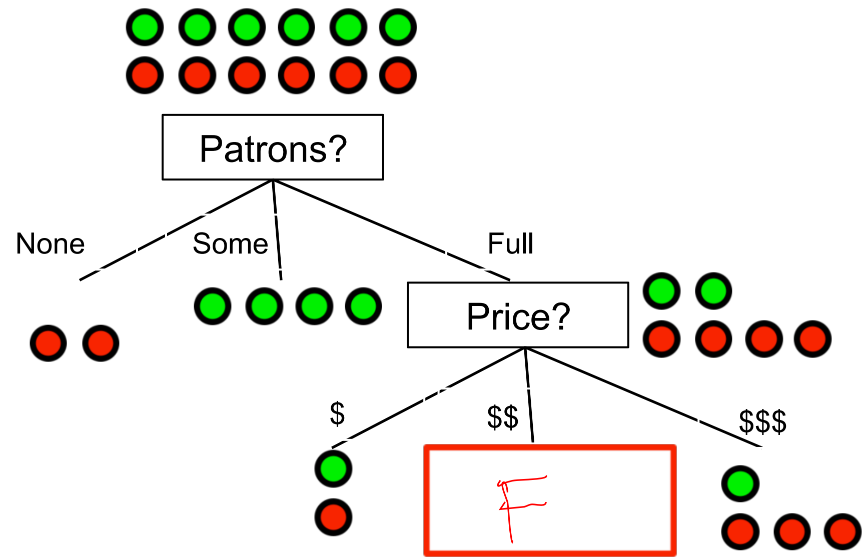
- Keep splitting until ….
  - All examples have same label
  - Run out of examples
  - Run out of attributes

Base Case I



- All examples have same classification (e.g. true)
  - Return classification (e.g., true)

Base Case II



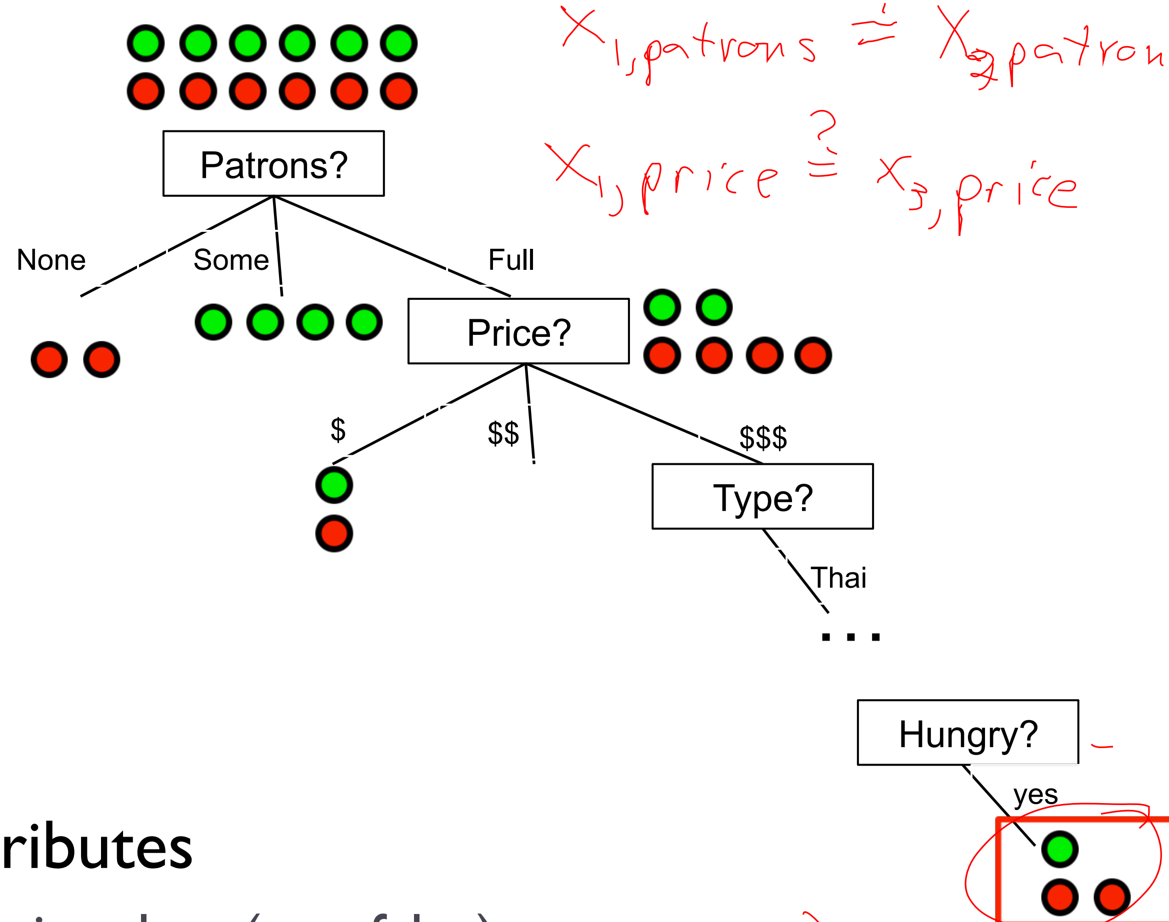
- No more examples
  - Return majority class of parent (e.g., false)

Base Case III

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$

$x_{1,patrons} \stackrel{?}{=} x_{2,patrons}$

$x_{1,price} \stackrel{?}{=} x_{3,price}$



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$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3$

- No more attributes
  - Return majority class (e.g., false)

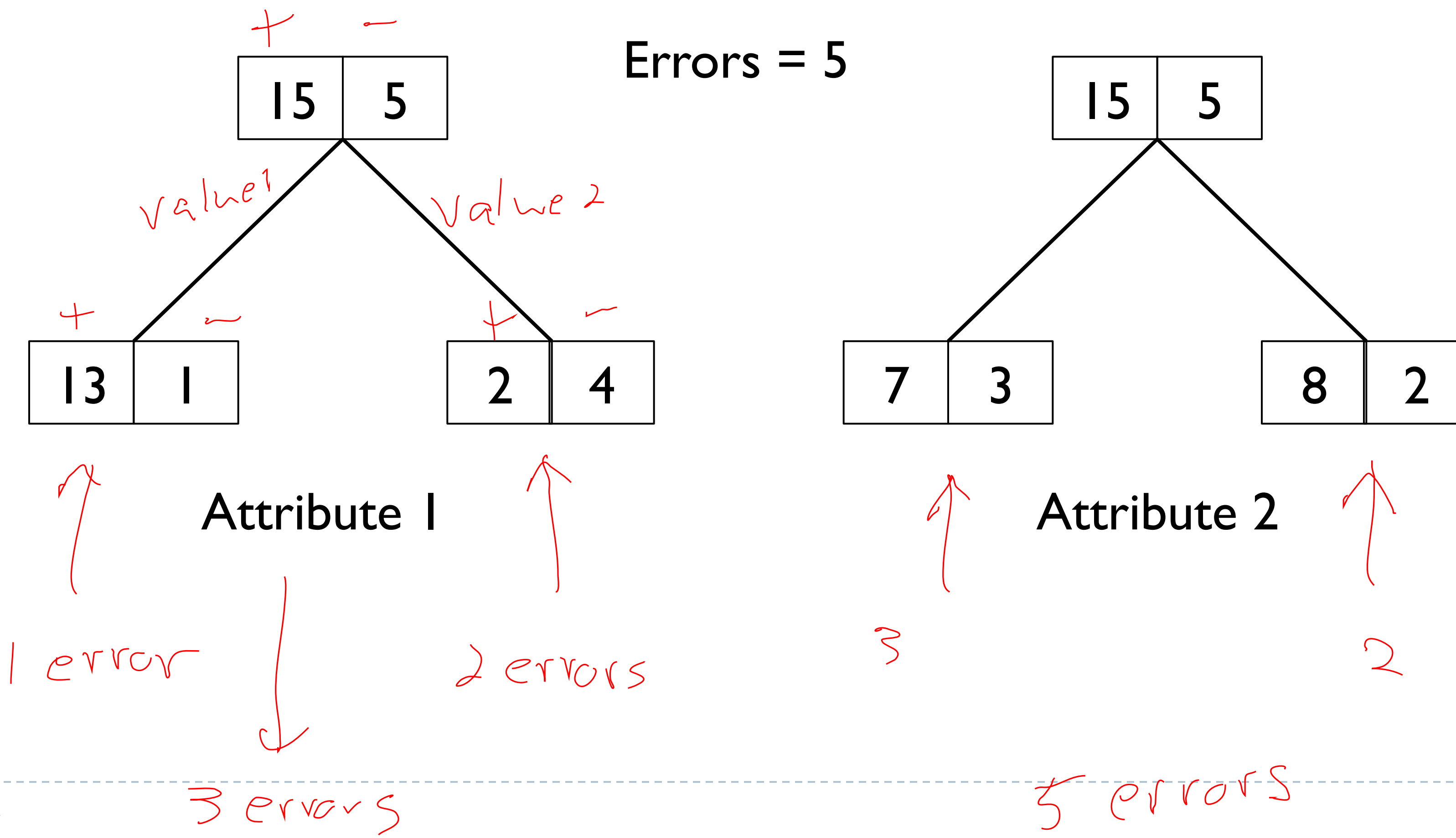
The Algorithm

- Put it (almost) all together

```
function DTL(examples, attributes, default) returns a decision tree
    if examples is empty then return default
    else if all examples have the same classification then return the classification
    else if attributes is empty then return MODE(examples)
    else
        best ← CHOOSE-ATTRIBUTE(attributes, examples)
        tree ← a new decision tree with root test best
        for each value v_i of best do
            examples_i ← {elements of examples with best = v_i}
            subtree ← DTL(examples_i, attributes − best, MODE(examples))
            add a branch to tree with label v_i and subtree subtree
        return tree
```

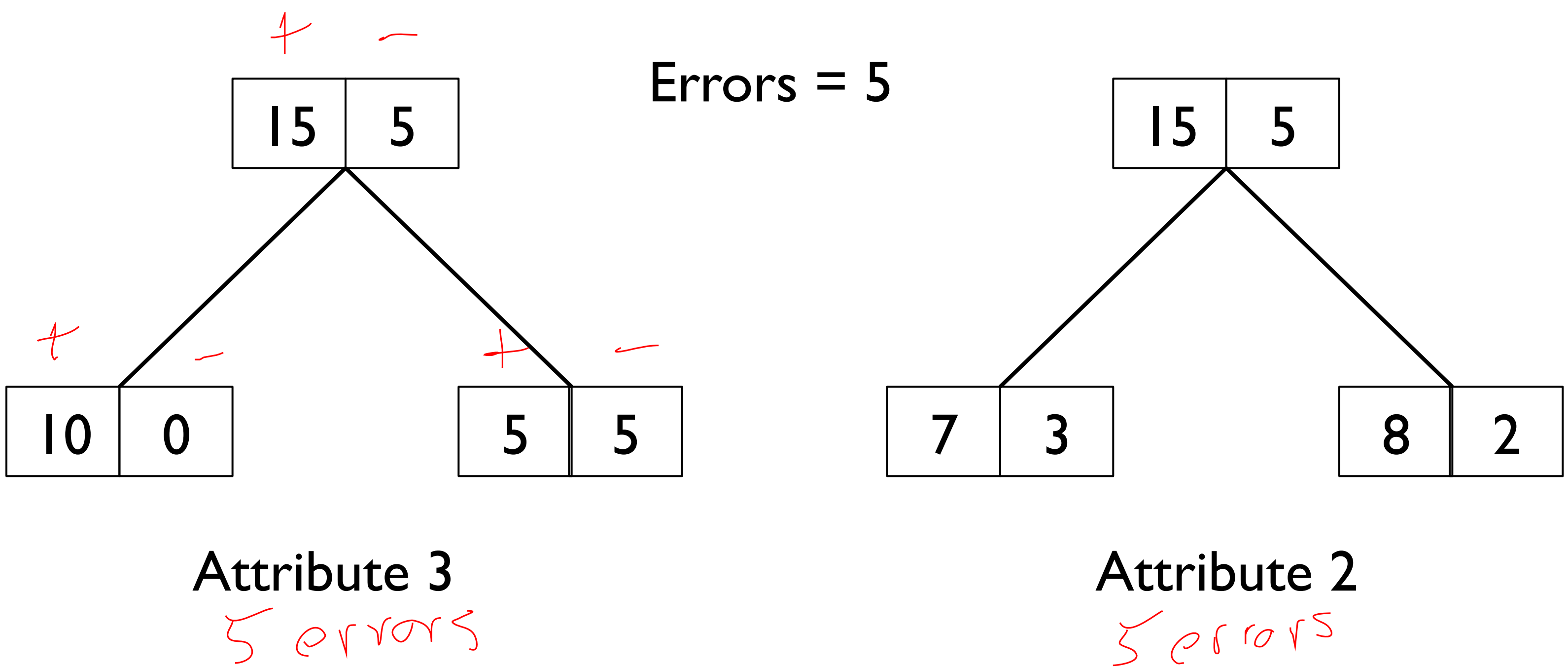
Splitting Criteria

- An obvious idea is error rate
  - Total # of errors we would make if we stopped training now



Splitting Criteria

- Problem: error rate fails to differentiate these two splits

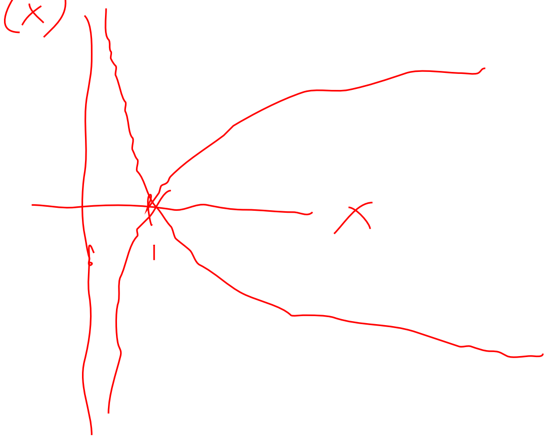


- HW problem: show that error rate does not decrease unless children have different majority classes word → codeword

Entropy

- Measure of uncertainty from field of Information Theory
  - Claude Shannon (1948)

$-\log(x)$



- Consider a biased random coin
  - $\Pr(X = \text{heads}) = q$ → 0.75
  - $\Pr(X = \text{tails}) = 1 - q$ → 0.25

- Define *surprise* of a random variable as
  - $S(X = x) = -\log_2 \Pr(X = x)$

$-\log(\frac{1}{4})$

$-\log_2(\frac{1}{2}) = -\log_2(2^{-1})$

	q = 0	q=1/4	q=1/2
S(X = heads)	infinite	2	1
S(X = tails)	0	.4150	1

$-\log(\frac{3}{4})$

#bits needed to represent a given word

Entropy

.81

	q = 0	q=1/4	q=1/2
S(X = heads)	infinite	1/4 × 2	1
S(X = tails)	0	3/4 × .4150	1

convention ⟹ $0 \times (-\log 0) + 1 \times (-\log 1)$ $\quad -\frac{1}{4}\log\left(\frac{1}{4}\right) - \frac{3}{4}\log\left(\frac{3}{4}\right)$

$= 0 + 0$ $\quad \approx .81$

- Entropy = average surprise

$$H(X) = -\sum_{\text{values } x} \Pr(X = x)\left(\log \Pr(X = x)\right)$$

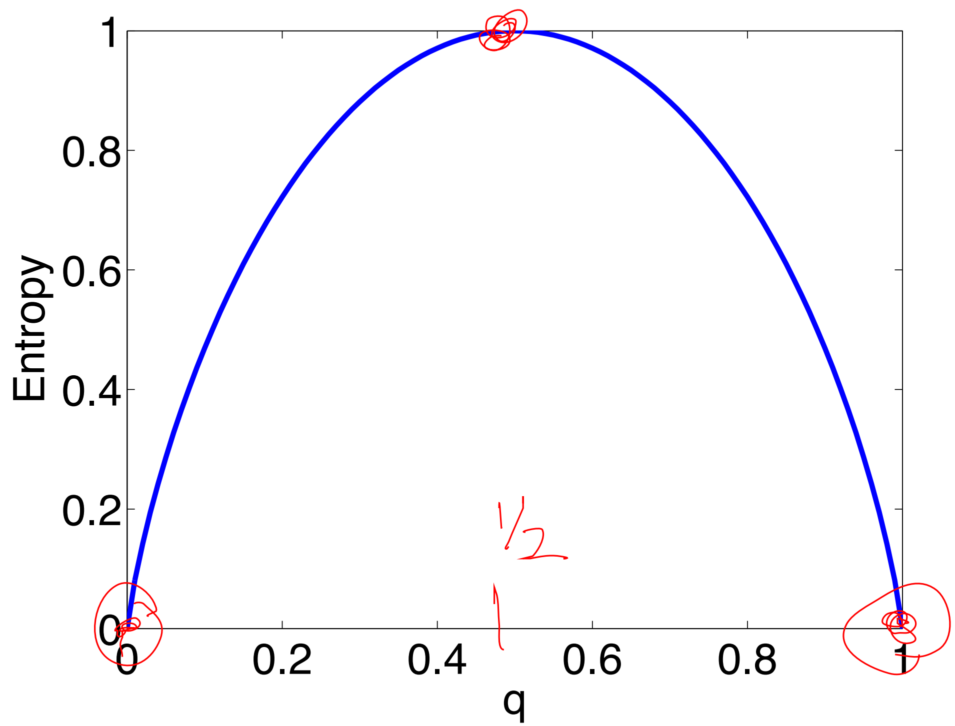
- For the biased coin

heads tails

$$\boxed{B(q) := H(X) = -q\log q - (1-q)\log(1-q)}$$

Entropy

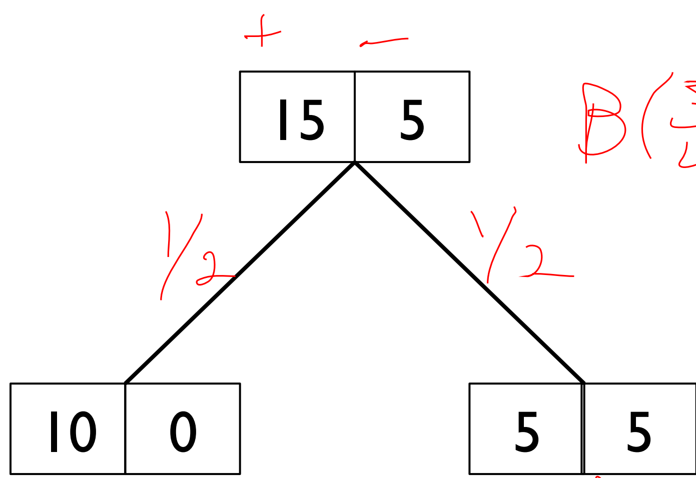
- Entropy of a biased coin



Entropy as Attribute Test

- Look for split with biggest reduction in entropy
  - *Information gain* = entropy – (average entropy of children)

q = fraction of positives



$B\left(\frac{3}{4}\right) = .81$

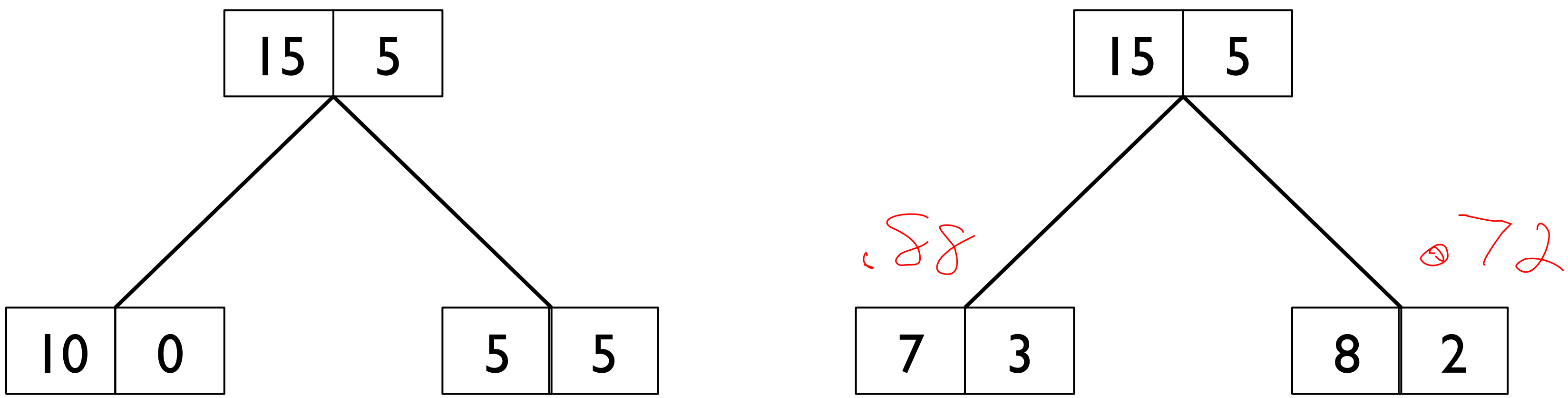
$B(1) = 0$

$B\left(\frac{1}{2}\right) = 1$

average entropy $= \frac{1}{2}B(1) + \frac{1}{2}B\left(\frac{1}{2}\right) = \frac{1}{2}$

Entropy as Attribute Test

- Back to our previous example



entropy $= \frac{1}{2}$

$B\left(\frac{7}{10}\right) = 0.88$

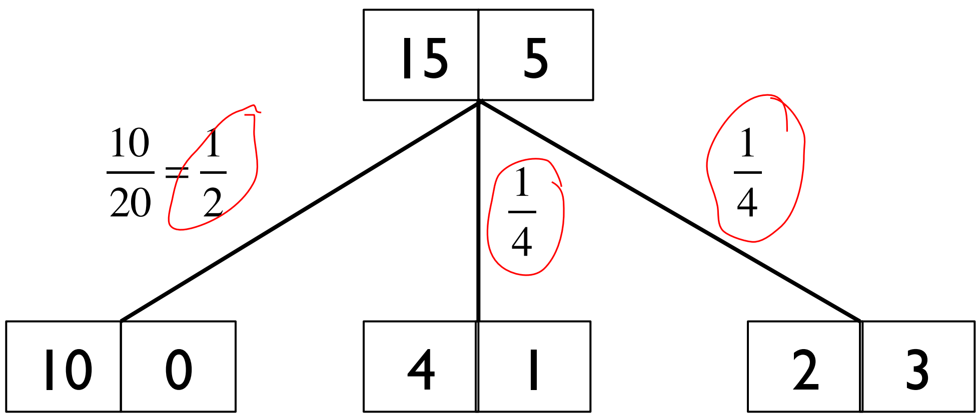
$B\left(\frac{8}{10}\right) = .72$

entropy $= .80$

$\text{average} = \frac{1}{2}B(1) + \frac{1}{4}B(\frac{4}{5}) + \frac{1}{4}B(\frac{2}{5})$

A More General Example

$$B(\tfrac{3}{4}) = -\tfrac{3}{4}\log\tfrac{3}{4} - \tfrac{1}{4}\log\tfrac{1}{4} = 0.88$$



$$B(1) = 0$$

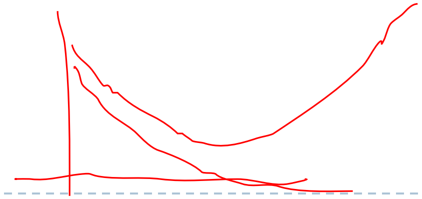
$$B(\tfrac{2}{5}) = -\tfrac{2}{5}\log\tfrac{2}{5} - \tfrac{3}{5}\log\tfrac{3}{5} = 0.97$$

$$B(\tfrac{4}{5}) = -\tfrac{4}{5}\log\tfrac{4}{5} - \tfrac{1}{5}\log\tfrac{1}{5} = 0.72$$

Average entropy of children $\quad \tfrac{1}{2}\cdot 0 + \tfrac{1}{4}\cdot 0.72 + \tfrac{1}{4}\cdot 0.97 = 0.42$

Reduction $\quad 0.88 - 0.42 = 0.46$

Pruning



- It is still possible to grow very large trees → overfitting

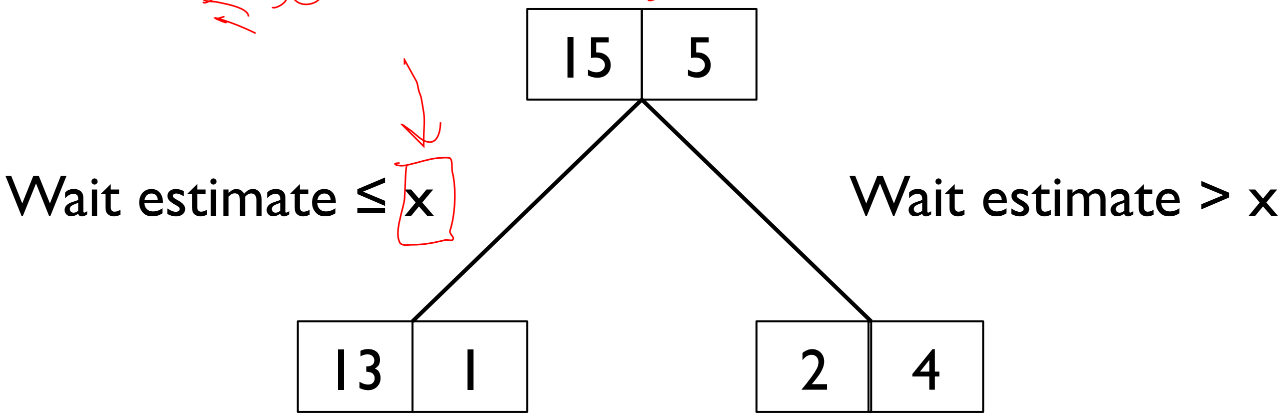
- Pruning
  - First grow a full tree
  - Then remove some nodes
  - (Why not just stop growing the tree earlier?)

- Different pruning criteria
  - R&N: Chi-squared. How "different' are the children from the parents?
  - An easier possibility: use a test set to see if removing the node hurts generalization

Continuous Attributes

wait ≤ 10 wait > 10

≤ 30 > 30



- Split on any threshold value *x*
  - Select attribute/threshold combination with best reduction in entropy

Miscellaneous

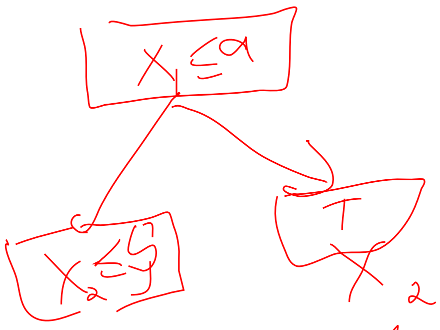
- Issues to think/read about on your own

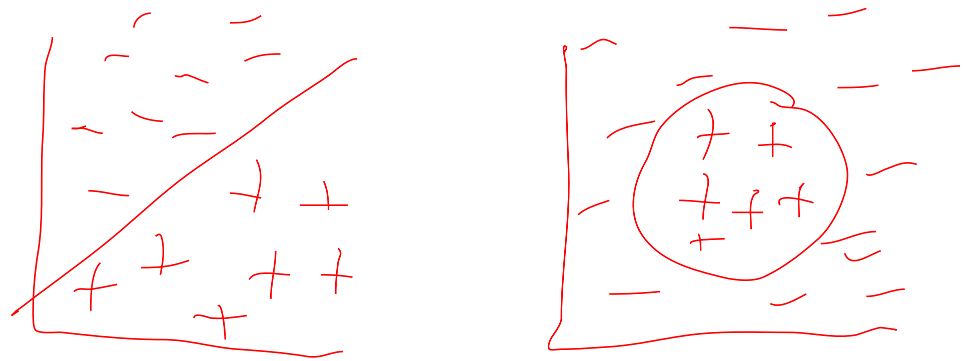
- Entropy criteria tends to favor multi-valued attributes over binary attributes
  - Why?
  - Various fixes proposed in literature. Can you think of one?

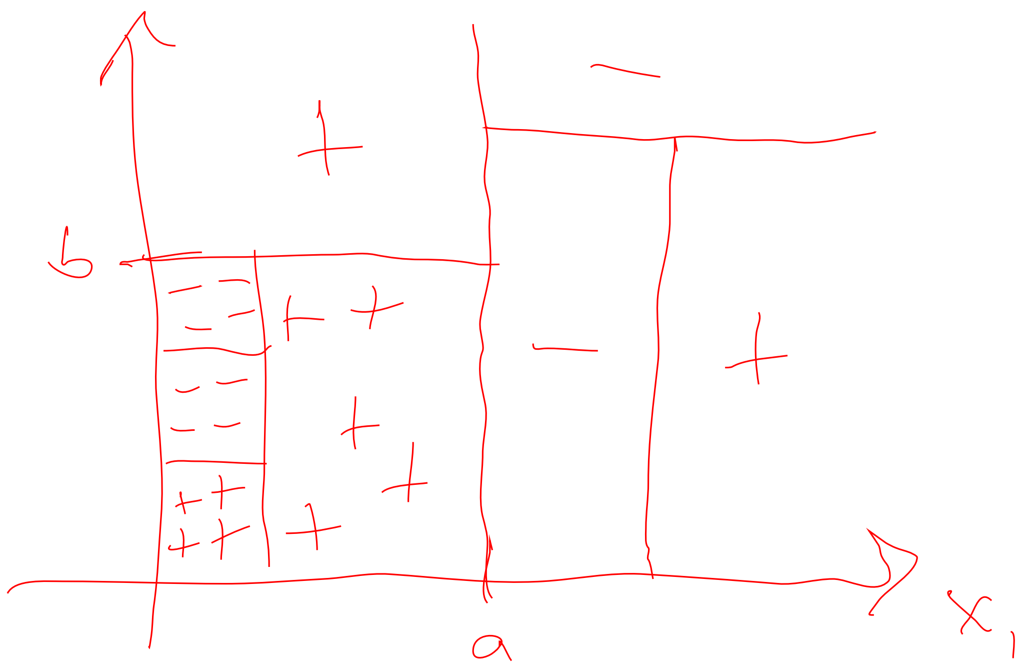
- Missing attributes
  - E.g., medical testing
  - How to handle these during prediction?
  - During training?

Geometric Interpretation

- Axis-parallel splits

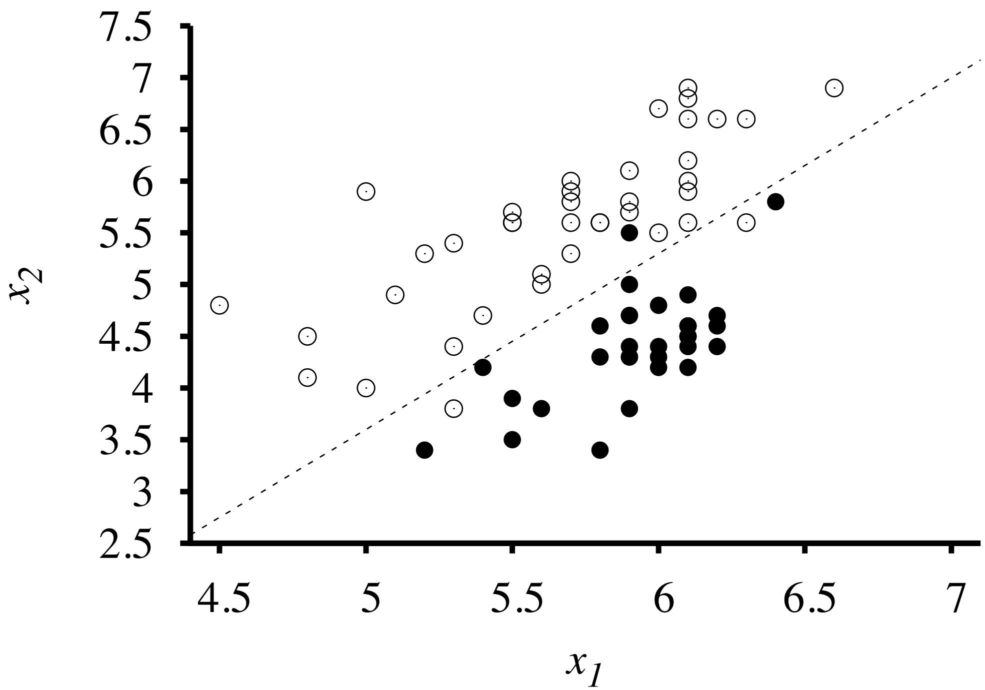


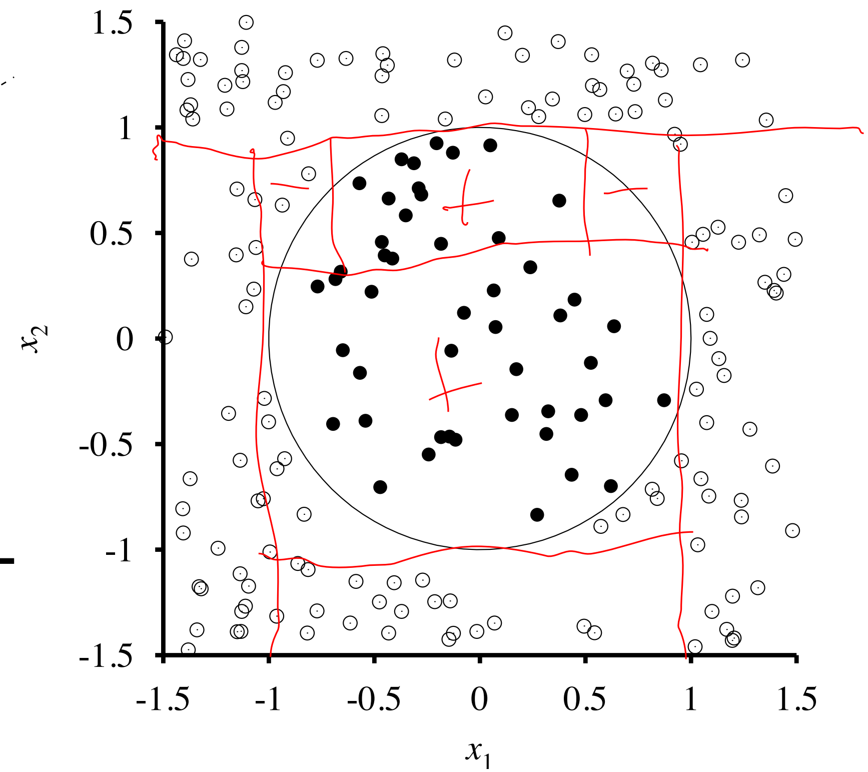




Geometric Interpretation

- Can learn simple and complex boundaries





What You Should Know

- Top-down greedy learning for decision trees
- Entropy splitting criteria
- Continuous attributes