Review of Derivatives

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Motivation

Functions of one or more variables

- $f(x) = (5x - 4)^2$
- $g(x, y) = 4x^2 - xy + 2y^2 - x - y$

- **Optimization**: find inputs that lead to smallest (or largest) outputs
  - value of x with smallest $f(x)$
  - (x, y) pair with smallest $g(x, y)$

Derivative

- Function $f : \mathbb{R} \to \mathbb{R}$
- Derivative $\frac{d}{dx} f(x)$
- (Also $f'(x)$, but we usually prefer the other notation)

Interpretation

- Slope of tangent line at x
- Illustration: function, tangent line, rise over run
- Rate of change

$$f(a+\epsilon) \approx f(a) + \epsilon \frac{d}{dx} f(a)$$

Optimization!

- If x is a maximum or minimum of f, then the derivative is zero

$$\frac{d}{dx} f(x) = 0$$

- Illustration: minimum, maximum, inflection point
- So, one way to *find* maximum or minimum is to set the derivative equal to zero and solve the resulting equaiton for x
  - Need an expression for $\frac{d}{dx} f(x)$

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- Chain rule
  - $f(g(x))' = f'(g(x)) \cdot g'(x)$
  - $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$
  - $\frac{d}{dx}f(g(x)) = \frac{df}{dg(x)} \cdot \frac{d}{dx}g(x)$

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Rules

- $\frac{d}{dx} \log x = \frac{1}{x}$
- $\frac{d}{dx} e^x = e^x$
- Quotient rule, product rule, etc.
- Many good references online

Example

$$\frac{d}{dx}4x^3 =$$

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&= 2\cdot(5x-4)\cdot(5-0) && \text{(polynomial)}\\
&= 10\cdot(5x-4)\\
&= 50x-40
\end{aligned}
$$

Exercises

Take derivative of same function, but first multiply out the quadratic:

$$\frac{d}{dx}(5x-4)^2 =$$

Back to optimization

- Our function and derivative:

$$f(x) = (5x - 4)^2 \quad \Longrightarrow \quad \frac{d}{dx} f(x) = 50x - 40$$

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- Our function and derivative:

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- Set equal to zero:

$$0 = \frac{d}{dx} f(x) = 50x - 40$$

Back to optimization

- Our function and derivative:

$$f(x) = (5x-4)^2 \quad \Longrightarrow \quad \frac{d}{dx} f(x) = 50x - 40$$

- Set equal to zero:

$$0 = \frac{d}{dx} f(x) = 50x - 40$$

- Solve:

$$x = 4/5$$

Back to optimization

- Our function and derivative:

$$f(x) = (5x - 4)^2 \implies \frac{d}{dx} f(x) = 50x - 40$$

- Set equal to zero:

$$0 = \frac{d}{dx} f(x) = 50x - 40$$

- Solve:

$$x = 4/5$$

Convex functions

- Is $x = 4/5$ a minimum, maximum, or inflection point?
- Illustration: convex / concave functions
  - Convex $=$ bowl-shaped
- Second derivative
  - $\frac{d^2}{dx^2} f(x) := \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = f''(x)$
- A function is convex if $\frac{d^2}{dx^2} f(x) \geq 0$ for all x
  - $\frac{d}{dx} f(a) = 0$ implies that a is a minimum

Wolfram Alpha

- Wolfram Alpha: http://www.wolframalpha.com/

(Optional Exercises)

- $\frac{d}{dx}\sqrt{x}$
- $\frac{d}{dx}3e^{4x}$

Wrap-up

- What to know
  - Intuition of derivative
  - How to take derivatives of simple functions
  - Convex, concave
  - Find minimum by setting derivative equal to zero and solving (for convex functions)
- Resources
  - Lots of material online
  - Wolfram Alpha: `http://www.wolframalpha.com/`
  - Mathematica, Maple, etc.