CS312: Top-Trading Cycles

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Alvin Roth and Lloyd Shapley won the 2012 Nobel Prize in Economics for their work on matching markets, which includes both the Top-Trading Cycle and Stable Matching problems. These notes are about Top-Trading Cycles, which were introduced by Alvin Roth and Herb Scarf (another Nobel laureate) in a 1962 paper. Roth and Scarf attribute the algorithm to David Gale.

Candy market setup:

- n people numbered $1, 2, 3, \ldots, n$
- n items of candy $c_1, c_2, \ldots, c_n$
- Person i initially has candy c_i
- Every person has a *complete ranked list of preferences* of candy items (no ties allowed)

An *allocation* is an assignment of one piece of candy to each person. In the initial allocation is the person i has item c_i.

The high-level goal of the problem is to make trades to find better allocations, i.e., ones where individuals get items that they like better. We formalize this by saying that we want a *stable allocation.* To define a stable allocation, let us first define what it means *not* to be stable.

Definition 1. *Given any allocation, say that the group of people S forms a* trading coalition *(with respect to the allocation), if they can swap items amongst themselves so every person receives an item that they like better.*

Definition 2. *An allocation is* stable *if there is no trading coalition.*

We now develop an algorithm to find a stable allocation.

Algorithm

- While people remain
  1. Everyone points to the owner of their favorite remaining candy item
  2. Find all cycles (a cycle is a sequence of people $p_1, p_2, \ldots, p_k$ such that p_i points to p_{i+1} for all $i \in \{1, \ldots k-1\}$, and the first and last person in the sequence are the same ($p_1 = p_k$)).
  3. For each cycle, make the trades suggested by the cycle: each person receives the item currently held by the person they are point to, which is their favorite remaining item. Remove those people and items from the game.

Correctness

Claim 1. *The top-trading cycle algorithm terminates.*

Proof. Note that at least one cycle is found each time through the while loop. We can see this by starting at any person p_1 and "following pointers", i.e., looking at the sequence of people $p_1, p_2, p_3, \ldots$ where person p_{i+1} is the unique person pointed to by p_i. Since everyone person points to one other person, we can keep following pointers and this sequence continues forever. Hence it must repeat at some point. In other words, there are two numbers $k < j$ such that $p_k = p_j$, so the subsequence $p_k, p_{k+1}, \ldots, p_j$ defines a cycle.

Hence, we remove at least one person each time through the loop. Since there are n people to begin with, the algorithm terminates in at most n passes through the loop. □

Claim 2. *The top-trading cycle algorithm terminates with a stable allocation.*

Proof. For the sake of contradiction, suppose there is some trading coalition S with respect to the final allocation.

Let N_1 be the set of people removed in the first stage. No one in N_1 can be part of S, because they all received their favorite items, so they cannot improve by any swap.

Now, let N_2 be the set of people removed in the second stage. Each of these people gets their favorite item not owned by anyone in N_1. So, the only way they could improve is by trading with someone in N_1, but since none of those people belong to S, they cannot improve by trading with anyone in S.

This argument can be repeated for people in N_3, N_4, etc. to show that no one can belong to S.

Thus, the assumption that there is a trading coalition is false, which means the allocation is stable. □