Consider an exponential family on $x_1, x_2 \in \{0, 1\}$ with $T(x_1, x_2) = I[x_1 = 1, x_2 = 1]$.
Suppose you use the data below to estimate maximum likelihood parameters:

$$\begin{array}{c|c|c|c|c|c}
  x_1 & x_2 & \text{data exp.} & \text{model exp.} & E_x[I] & E_{\theta^*}[E_x[I|x_1, x_2]] \\
  \hline
  1 & 1 & \exp(T(x)) & \exp(T(\theta)) & I_2 & I_2 \\
  1 & 0 & & & & \\
  0 & 1 & & & & 
\end{array}$$

At the maximum likelihood estimate $\theta^*$, what will be $P_{\theta^*}(X_1 = 1, X_2 = 1)$? $\sqrt{2}$
Monte Carlo Methods

Motivation

Computing expectations is important!

\[ \mathbb{E}_{p(x)}[f(X)] = \int p(x)f(x)dx \]

Example: suppose \( p(x) \) is an MRF, then

\[ P(X_u = a, X_v = b) = \mathbb{E}_{p(x)}[1[X_u = a, X_v = b]] \]

In general, computing expectations is hard, so we need an approximation.

Monte Carlo methods

In a Monte Carlo method, we approximate an expected value by a sample average. Draw \( N \) samples \( X_1, \ldots, X_N \sim p(x) \), then

\[ \mathbb{E}_{p(x)}[f(X)] \approx \frac{1}{N} \sum_{n=1}^{N} f(X_n). \]

Nice properties:

- Unbiased
- Variance decreases like \( \frac{1}{N} \).
- Measure arbitrary properties by choosing \( f \).

Not nice properties: sampling is algorithmically/computationally hard in general.

Examples

Suppose we have \( p(x) = 12(x^2 - x^3) \), where \( x \in [0, 1] \). Or suppose we have an MRF with a cycle.

Question: How do we sample from these distributions? Answer: some algorithm.
Gibbs Sampling

Markov Chain Monte Carlo Overview

- Markov chain Monte Carlo (MCMC) methods iteratively construct samples from a given target distribution $p(x)$.
- They require only access to the unnormalized distribution, so can apply easily to models like MRFs.
- Formally, they work by constructing a Markov chain that has the target distribution $p(x)$ as its limiting distribution.
- We'll introduce one MCMC method today, and then start to develop some of the theory needed to understand the algorithm.
- Importance / applications: statistical physics, econometrics, ecology, epidemiology, weather modeling, . . .

The Gibbs Sampler

A simple and powerful algorithm! Assume $X = (X_1, \ldots, X_D)$.

Initialize all variables arbitrarily, then repeatedly update each variable by sampling from its conditional distribution given all other variables.

Gibbs sampler

- Initialize $x_1, \ldots, x_D$
- Repeat
  - For $i = 1$ to $D$, resample $x_i \sim p(X_i | X_{-i} = x_{-i})$
  - Record $x = (x_1, \ldots, x_D)$ as one sample

One sample is generated after each loop through all of the variables.

Example: Cycle MRF

Suppose $p(x) = \prod_{i=1}^{n-1} \phi(x_i, x_{i+1}) \bmod n$

Update $x_2$

$p(x_2 | x_1, x_3, x_4, \ldots, x_n) \propto \phi(x_2, x_3) \phi(x_3, x_4) \cdots \phi(x_n, x_1)$

Then $p(x_2 | x_{-2}) \propto \phi(x_{n-1}, x_1) \phi(x_1, x_2) \phi(x_2, x_3) \phi(x_3, x_4) \cdots \phi(x_{n-2}, x_{n-1})$ (factor reduction!)

For a general MRF: $p(x_2 | x_{-2}) \propto \prod_{i \in c_2} \phi_i(x_2, x_{c_2 \backslash i})$
The Gibbs Sampler: Properties

- The Gibbs sampler eventually draws samples from the target distribution \( p(x) \) regardless of how it is initialized.
- It can take time to converge to the target distribution \( p(x) \). This phase of the algorithm is referred to as the "burn-in" phase of the algorithm.
- Convergence to the target distribution needs to be tested empirically in most cases using convergence diagnostics.
- Even after convergence, the samples are not independent, but can still be used in Monte Carlo averages. The degree of correlation of the samples affects the rate of convergence of Monte Carlo averages.

Markov Chains

A discrete Markov chain is a set of states with transition probabilities between each pair of states. **Example** (note: not a graphical model!)

Transition Matrix

- The probabilistic transitions in the state diagram can also be represented by an equivalent matrix of transition probabilities.
- The "from" states are rows and the "to" states are columns.
Monte Carlo Methods
Gibbs Sampling
Markov Chain Theory

**Markov Chains: Simulation and State Sequences**

- To simulate a Markov chain, we draw $x_0 \sim p_0$, then repeatedly sample $x_{t+1}$ given the current state $x_t$ according to the transition probabilities $T$.

**Markov Chain: Formal Definition**

By repeatedly making random transitions from a starting state, we generate a chain of random variables $X_0, X_1, X_2, X_3, \ldots$.

Formally, a Markov chain is specified by:
- A set of states $\{1, 2, \ldots, D\}$
- A starting distribution $p_0$ with $p_0(i) = P(X_0 = i)$.
- Transition probabilities $T_{ij} = P(X_{t+1} = j \mid X_t = i)$ for all $i, j \in \{1, 2, \ldots, D\}$

A Markov chain assumes the Markov property:

$$P(X_t = x_t \mid X_0 = x_0, X_1 = x_1, \ldots, X_{t-1} = x_{t-1}) = P(X_t = x_t \mid X_{t-1} = x_{t-1})$$

**Markov Chain Questions**

Three important questions:
1. What is the joint probability of a sequence of states of length $N$?
2. What is the marginal probability distribution over states after a given number of steps $t$?
3. What happens to the probability distribution over states in the limit as $t$ goes to infinity?

**Markov Chain Factorization**

**Question:** What is the joint probability over the state sequence $x_0, \ldots, x_N$?

**Answer:** by the Markov property:

$$P(X_1 = x_1, \ldots, X_N = x_N \mid X_0 = x_0) = P(X_1 = x_1 \mid X_0 = x_0) \times P(X_2 = x_2 \mid X_1 = x_1) \times \cdots \times P(X_N = x_N \mid X_{N-1} = x_{N-1})$$

Shorter version:

$$p(x_1, x_2, \ldots, x_N \mid x_0) = p(x_1 \mid x_0) p(x_2 \mid x_1) \cdots p(x_N \mid x_{N-1}) = T_{x_0 x_1} \times T_{x_1 x_2} \times \cdots \times T_{x_{N-1} x_N}$$
The \(t\)-Step Distribution for Fixed \(x_0\)

**Question:** What is the marginal probability distribution after \(t\) steps given that the chain starts at \(x_0\)? I.e., what is \(p(x_t|x_0)\)?

**Examples:**

\[
p(x_1|x_0) = \\
p(x_2|x_0) = \\
\]

In general, we have the recursive expression:

\[
p(x_t|x_0) = \sum_{x_{t=1}} p(x_t|x_{t=1},x_t) = \sum_{x_{t=1}} p(x_t|x_{t=1}) T x_{t=1} x_t.
\]

### \(t\)-Step Recurrence as Matrix-Vector Multiplication

The recurrences for the \(t\)-step distributions can be expressed using matrix-vector multiplication. Let \(p_t\) be the row-vector

\[
p_t = [P(X_t = 1), P(X_t = 2), \ldots, P(X_t = D)].
\]

Then, since \(T_{ij} = P(X_t = j|X_{t-1} = i)\), we can write the above recursive relationship as

\[
p_t = p_{t-1} T.
\]

---

The \(t\)-Step Distribution for Random \(X_0\)

**Question:** What is the marginal probability distribution after \(t\) steps given that \(X_0 \sim p_0\)? I.e., what is \(p(x_t)\)?

By similar logic:

\[
p(x_1) = \\
p(x_2) = \\
\]

In general:

\[
p(x_t) = 
\]
t-Step Distribution as Matrix Power

By unrolling the recurrence, the $t$-step distribution can be obtained as a matrix power

$$p_t = p_0T^t$$

$$= (p_{t-1})T$$

$$= (p_{t-2})TT$$

$$= (p_{t-3})TTT$$

$$\vdots$$

$$= p_0 T \cdots T, \quad \text{t times}$$

Thus

$$p_t = p_0 T^t.$$ 

This also implies that $T^t$ is the $t$-step transition matrix

$$(T^t)_{ij} = P(X_t = j | X_0 = i) = P(X_{s+t} = j | X_s = i).$$