CS 312: Algorithms **Fall 2018**

Fourth Hour 2

Your Name: ________________ Collaborators: ____________________________

Problem 1. Big-O Exercises

1. Let $T(n) = 4n^3 + 2n^2 + 2$. Prove that $T(n)$ is $O(n^3)$ using only the definition of big-O (i.e., do not use any of the properties we proved about big-O such as transitivity or additivity).
2. Let $T(n) = n \log(n)$. Prove that $T(n)$ is $O(n^2)$.
3. Let $T(n) = n \log(n)$. Prove that $T(n)$ is *not* $O(n)$.

Problem 2. Asymptotics: K&T Ch. 2 Ex. 5 Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give a proof or counterexample.

- $f(n)^2$ is $O(g(n)^2)$
- $2^{f(n)}$ is $O(2^{g(n)})$

Problem 3. Asymptotics. Take the following list of functions and arrange them in ascending order of growth rate. That is, if function $g(n)$ immediately follows function $f(n)$ in your list then it should be the case that $f(n)$ is $\mathcal{O}(g(n))$.

$$f_1(n) = 10^n$$
$$f_2(n) = n^{1/3}$$
$$f_3(n) = n^n$$
$$f_4(n) = \log_2 n$$
$$f_5(n) = 2^{\sqrt{\log_2 n}}$$

Problem 4. Stable Matchings: K&T Ch 1, Ex 5. *If you have time remaining, start working on this homework problem.*

Consider a version of the stable matching problem where there are n students and n colleges as before. Assume each student ranks the colleges (and vice versa), but now we allow ties in the ranking. In other words, we could have a school that is indifferent two students s_1 and s_2, but prefers either of them over some other student s_3 (and vice versa). We say a student s *prefers* college c_1 to c_2 if c_1 is ranked higher on the s's preference list and c_1 and c_2 are not tied.

1. **Strong Instability.** A strong instability in a matching is a student-college pair, each of which prefer each other to their current pairing. In other words, neither is indifferent about the switch. Does there always exist a matching with no strong instability? Either give an example of a set of colleges and students with preference lists for which every perfect matchings has a strong instability; or give an algorithm that is guaranteed to find a matching with no strong instability and prove that it is correct.

2. **Weak Instability.** A weak instability in a matching is a student-college pair where one party prefers the other, and the other may be indifferent. Formally, a student s and a college c with pairs c' and s' form a weak instability if either

   - s prefers c to c' and c either prefers s to s' or is indifferent between s and s'.
   - c prefers s to s' and s either prefers c to c' or is indifferent between c and c'.

   In other words, the pairing between c and s is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of colleges and students with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.