COMPSCI 311: Introduction to Algorithms

Lecture 24: More NP-Complete Problems

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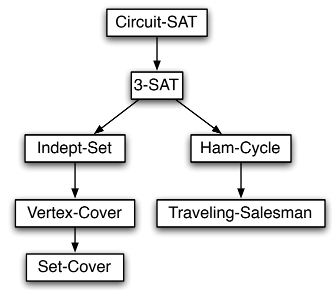
NP-Complete Problems So Far



Arrows show reductions discussed in class.
We could construct a polynomial reduction between any pair.

NP-Completeness and Reductions

Careful, direction of reduction matters!

A $\leq_P$ B: A reduces **to** B (A "no harder" than B)
From arbitrary instance of A, construct instance of B
Reduction and construction is **one-way**

Problem instances are **equivalent** (**both ways**):
$\text{YES}_A \implies \text{YES}_B$
$\text{YES}_B \implies \text{YES}_A$ (same as $\text{NO}_A \implies \text{NO}_B$)

B is NP-complete means:

1. B is in NP: can **check** solution in polynomial time ("easy enough")
2. B is NP-hard: some NP-complete A reduces **to** B: A $\leq_P$ B ("hard enough"). We also say: reduce **from** A.

Clicker

Which of the following graph problems are in NP?

A. Length of longest simple path is $\leq k$

B. Length of longest simple path is $= k$

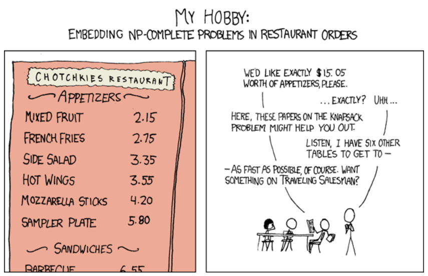
C. Length of longest simple path is $\geq k$

D. Find length of longest simple path.

E. All of the above.

Numerical problems

Subset Sum decision problem: given n items with weights $w_1, \ldots, w_n$, is there a subset of items whose weight is exactly W?



Dynamic programming: $O(nW)$ *pseudo-polynomial* time algorithm (not polynomial in input length $n \log W$)

Subset Sum Warmup

Does this instance have a solution?

```
w1  1010
w2  1001
w3  0110
w4  0101
    ----
 W  1111
```

A. Yes
B. No

Subset Sum Warmup

For which nonzero values of y does this instance have a solution?

```
10010
10001
01001
01010
00111
00100
-----
1113y
```

A. $y = 1$
B. $y = 1, 2$
C. $y = 1, 2, 3$

Subset Sum Warmup

For which nonzero values of y does this instance have a solution?

```
10010
10011
01001
01000
00111
00100
-----
1112y
```

A. $y = 1$
B. $y = 1, 2$
C. $y = 1, 2, 3$

Subset Sum

Theorem. Subset sum is NP-complete.

Reduction from 3-SAT. (n variables, m clauses).

Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

	variable digits			clause digits
Item	1	2	3	
t_1	1	0	0	
f_1	1	0	0	
t_2	0	1	0	
f_2	0	1	0	
t_3	0	0	1	
f_3	0	0	1	
W	1	1	1	

- Items t_i, f_i for each x_i; correspond to truth assignment
- Weights $\implies$ select exactly one
- (Numbers are base 10)

Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

	variable digits			clause digits		
Item	1	2	3	4	5	6
t_1	1	0	0	1	0	0
f_1	1	0	0	0	1	1
t_2	0	1	0	0	1	0
f_2	0	1	0	1	0	1
t_3	0	0	1	1	0	1
f_3	0	0	1	0	1	0
W	1	1	1	?	?	?

- Clause digit equal to 1 iff x_i assignment satisfies C_j
- Total for clause digit > 0 iff assignment satisfies C_j
- Goal: all clause digits > 0. How to set W to enforce this? Total could be 1, 2, 3 for satisfied clause.

Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

	variable digits			clause digits		
Item	1	2	3	4	5	6
t_1	1	0	0	1	0	0
f_1	1	0	0	0	1	1
t_2	0	1	0	0	1	0
f_2	0	1	0	1	0	1
t_3	0	0	1	1	0	1
f_3	0	0	1	0	1	0
W	1	1	1	3	3	3

- Set all clause digits of W to 3... then add dummy items to increase total by **at most two**.

Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

	variable digits			clause digits		
Item	1	2	3	4	5	6
t_1	1	0	0	1	0	0
f_1	1	0	0	0	1	1
t_2	0	1	0	0	1	0
f_2	0	1	0	1	0	1
t_3	0	0	1	1	0	1
f_3	0	0	1	0	1	0
W	1	1	1	3	3	3

	variable digits			clause digits		
Item	1	2	3	4	5	6
y_1	0	0	0	1	0	0
z_1	0	0	0	1	0	0
y_2	0	0	0	0	1	0
z_2	0	0	0	0	1	0
y_3	0	0	0	0	0	1
z_3	0	0	0	0	0	1

- **Two** dummy items per clause $\Rightarrow$ can increase total by up to 2
- Can make total exactly 3 iff total of non-dummy items is > 0

Subset Sum Reduction: Details (Review on Own)

- All weights have $n + m$ digits
- For variable x_i, create two items t_i, f_i
  - Both have ith digit equal to 1
  - All other items have zero in this digit
  - ith digit of $W = 1 \Rightarrow$ select exactly one of t_i, f_i
- The $n + j$th digit corresponds to clause C_j
  - If $x_i \in C_j$, set $n + j$th digit of $t_i = 1$
  - If $\neg x_i \in C_j$, set $n + j$th digit of $f_i = 1$
  - Everything else 0.

- Set $n + j$th digit of $W = 3$
  - Consider a subset of items corresponding to a truth assignment (exactly one of t_i, f_i)
  - If C_j is not satisfied, then total in position $n + j$ is 0, otherwise it is 1, 2, or 3
  - Create two “dummy” items y_j, z_j with 1 in position $n + j$
  - Can select dummies to yield total of 3 in position $n + j$ iff C_j is satisfied

Subset Sum Proof

- All numbers (including W) are polynomially long.
- If Φ satisfiable,
  - Select t_i if $x_i = 1$ in satisfying assignment else select f_i.
  - Take y_j, z_j as needed.
- If subset exists with sum W
  - Either t_i or f_i is chosen. Assign x_i accordingly.
  - For each clause, at least one term must be selected, otherwise clause digit is < 3.

Graph Coloring

Def. A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, \ldots, k\}$ such that for all $(u, v) \in E$, $f(u) \neq f(v)$.

Problem. Given $G = (V, E)$ and number k, does G have a k-coloring?

Many applications

- Actually coloring maps!
- Scheduling jobs on machine with competing resources.
- Allocating variables to registers in a compiler.

Claim. $2\text{-}\textsc{coloring} \in$ P (equivalent to bipartite testing)

Theorem. $3\text{-}\textsc{coloring}$ is NP-Complete.

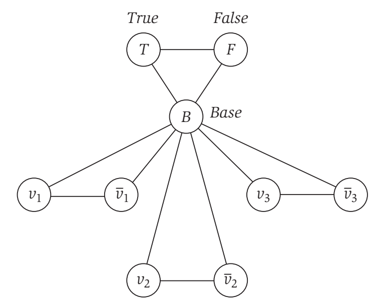
3-Color: Gadget for Variables

- Reduce from 3-SAT.

3 colors: True, False, "Base"

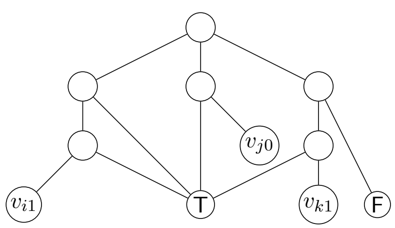
3 special nodes in a clique T, F, B.
For each variable x_i, two nodes v_{i0}, v_{i1}.
Edges $(v_{i0}, B), (v_{i1}, B), (v_{i0}, v_{i1})$.

Either v_{i0} or v_{i1} colored T, the other colored F.



Reduction: Clause Gadget

For clause $x_i \vee \neg x_j \vee x_k$



Top node can be colored iff not all three v-nodes are F.

Proof

- Graph is polynomial in $n + m$.
- If satisfying assignment
  - Color B, T, F then v_{i1} as T if $\phi(x_i) = 1$.
  - Since clauses satisfied, can color each gadget.
- If graph 3-colorable
  - One of v_{i0}, v_{i1} must get T color.
  - Clause gadget colorable iff clause satisfied.

Question. What about k-coloring?

Clicker Question

Which of the following is true?

A. If we can reduce 3-coloring to k-coloring, then k-coloring is NP-complete

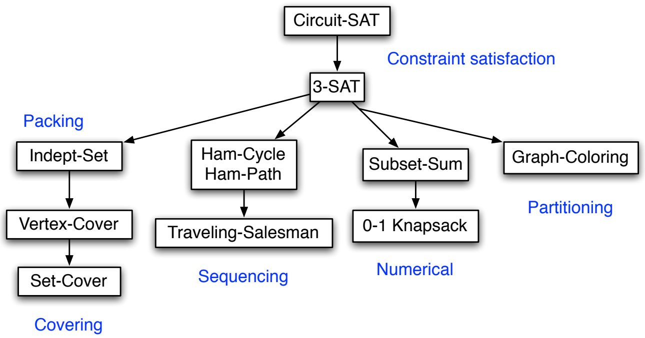
B. k-coloring is NP-complete since any 3-coloring is also a k-coloring for $k \geq 3$

C. k-coloring is not NP-complete since 3-coloring is the hardest case, for $k > 3$ the coloring is easier

D. k-coloring is not NP-complete because the 4-color theorem has been proved

NP-Completeness Recap

Types of hard problems:



. . . any many others. See book or other sources for more examples. You can use *any known NP-complete* problem to prove a new problem is NP-complete.