COMPSCI 311: Introduction to Algorithms

Lecture 23: Reductions and NP-Complete Problems

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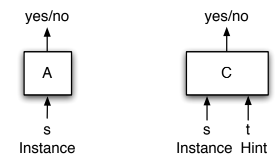
Review

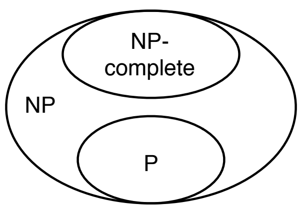
- P – class of problems with polytime **algorithm.**
- NP – class of problems with polytime **certifier**.



Example

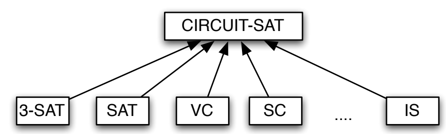
Problem (X)	INDEPENDENT-SET
Instance (s)	Graph G and number k
Algorithm (A)	No poly-time algorithm known
Hint (t)	Which nodes are in the answer?
Certifier (C)	Are those nodes independent and size k?

NP-Complete



- NP-complete = a problem $Y \in$ NP with the property that $X \leq_P Y$ for every problem $X \in$ NP!

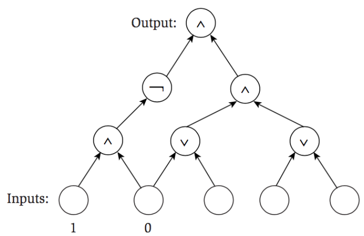
NP-Complete



- **Cook-Levin Theorem**: In 1971, Cook and Levin independently showed that particular problems were NP-Complete.
- We'll look at CIRCUIT-SAT as canonical NP-Complete problem.

Circuit-SAT

Problem: Given a circuit built of AND, OR, and NOT gates with some hard-coded inputs, is there a way to set remaining inputs so the output is 1?



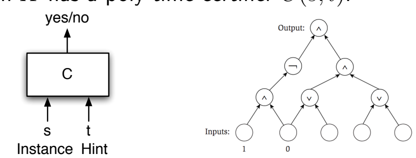
Satisfiable? Yes. Set inputs: 1, 1, 0.

Circuit-SAT

Cook-Levin Theorem Circuit-SAT is NP-Complete.

Proof Idea: encode arbitrary certifier $C(s,t)$ as a circuit

- If $X \in$ NP, then X has a poly-time certifier $C(s,t)$:



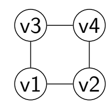
- s is Yes instance $\Leftrightarrow \exists\, t$ such that $C(s,t)$ outputs Yes
- Construct a circuit where s is hard-coded, and circuit is satsifiable iff $\exists\, t$ that causes $C(s,t)$ to output Yes
- s is Yes instance $\Leftrightarrow$ circuit is satisfiable
- Algorithm for Circuit-Sat implies an algorithm for X

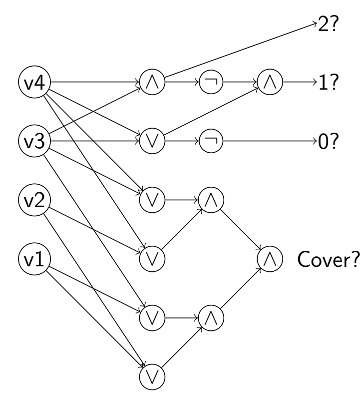
A Circuit-SAT reduction

See Independent Set example in other slides

A Circuit-SAT reduction

- Vertex Cover – Does G have VC of size at most k? (Counting gadget is an example for v_3, v_4 only)





Proving New Problems NP-Complete

Suppose X is in NP.

Fact: If Y is NP-complete and $Y \leq_P X$, then X is NP-complete.

Want to prove problem X is NP-complete

- Check $X \in$ NP.
- Choose known NP-complete problem Y.
- Prove $Y \leq_P X$.

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It's easy to show that $3\text{-SAT} \leq_P \textsc{Circuit-SAT}$. What can we conclude from this?

A. 3-SAT is NP-complete.

B. 3-SAT is in NP.

C. If there is no polynomial time algorithm for 3-SAT, then there is no polynomial time algorithm for $\textsc{Circuit-SAT}$.

Proving New Problems NP-Complete

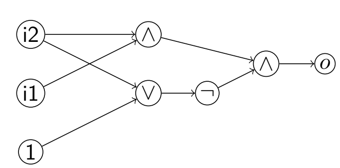
Theorem: 3-SAT is NP-Complete.

- In NP? Yes, check satisfying assignment in poly-time.
- Can show that $\textsc{Circuit-SAT} \leq_P 3\text{-SAT}$

From CIRCUIT-SAT to 3-SAT

To show that $\textsc{Circuit-SAT} \leq_P 3\text{-SAT}$, we'll show how to construct a 3-SAT formula to model an arbitrary $\textsc{Circuit-SAT}$ instance.

Example.



Reduction: Circuit-Sat $\leq_P$ 3-Sat

- One variable x_v per circuit node v plus clauses to enforce circuit computations
- Express Negation, OR, and AND nodes using several implications of the form $A \Rightarrow B$ (which is equivalent to the clause $\neg A \vee B$)

- Negation node: $x_v = \neg x_u$
  - $x_u \Rightarrow \neg x_v$
  - $\neg x_u \Rightarrow x_v$
- OR node: $x_v = x_u \vee x_w$
  - $x_u \Rightarrow x_v$
  - $x_w \Rightarrow x_v$
  - $x_v \Rightarrow x_u \vee x_w$
- AND node: $x_v = x_u \wedge x_w$
  - $x_v \Rightarrow x_u$
  - $x_v \Rightarrow x_w$
  - $\neg x_v \Rightarrow \neg x_u \vee \neg x_w$

Reduction: Circuit-Sat $\leq_P$ 3-Sat

- Clause $C = x_v$ for input bits v fixed to one
- Clause $C = \neg x_v$ for input bits v fixed to zero
- Clause $C = x_o$ for output bit

- This formula is satisfiable iff circuit is satisfiable.
- Deal with clauses of size 1 and 2 by introducing two new variables and clauses that force them to be equal to zero.

Clicker

Which of the following statements is NOT true?

A. SAT $\leq_P$ 3-SAT

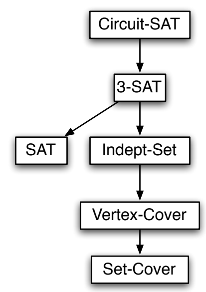
B. 3-SAT $\leq_P$ SAT

C. k-SAT $\leq_P$ SAT for all $k \geq 2$

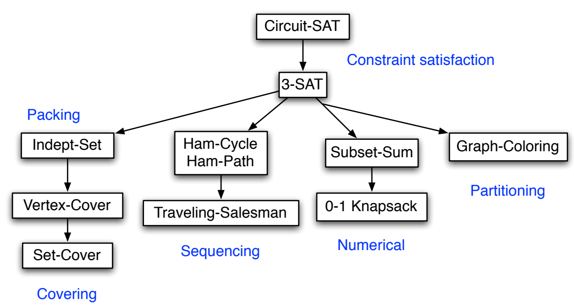
D. k-SAT is NP-complete for all $k \geq 2$

NP-Complete Problems So Far

Theorem: IndependentSet, VertexCover, SetCover, SAT, 3-SAT are all NP-Complete.



NP-Complete Problems: Preview

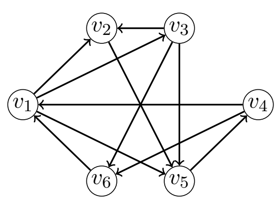


Traveling Salesman Problem

- TSP. Given n cities and distance function $d(i, j)$, is there a tour that visits all cities with total distance less than D?
  - Tour: ordering of cities $i_1, i_2, \ldots, i_n$ with $i_1 = 1$
  - Distance is $\sum_{j=1}^{n-1} d(i_j, i_{j+1}) + d(i_n, 1)$
- Applications: traveling salesman, moving robotic arms
- Let's prove a simpler problem is NP-complete, and then use it to show TSP is NP-complete.

Hamiltonian Cycle Problem

- HAMCYCLE – Hamiltonian Cycle. Given directed graph $G = (V, E)$, is there a cycle that visits each vertex exactly once?



- $v_1, v_3, v_2, v_5, v_4, v_6$ is a Hamiltonian Cycle

HAM-CYCLE

Theorem. HAM-CYCLE is NP-Complete.

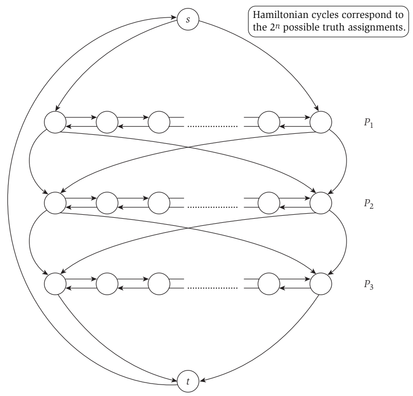
- It is in NP.
- Need to reduce from some NP-Complete problem. Which one?

Claim. 3-SAT $\leq_P$ HAM-CYCLE.

Reduction has two main parts.

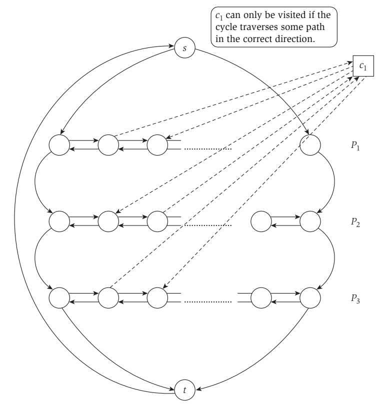
- Make a graph with 2^n Hamiltonian cycles, one per assignment.
- Augment graph with clauses to invalidate assignments.

Reduction: Graph skeleton



$x_i = 1 \iff$ traverse P_i from $L \to R$

Reduction: Clause Gadgets



$C_1 = x_1 \vee \bar{x}_2 \vee x_3$

Reduction: High-Level

- Correspondence between Hamiltonian cycles and truth assignments
  - $x_i = 1$: traverse path P_i from $L \to R$
  - $x_i = 0$: traverse path P_i from $R \to L$
- Node c_j for clause C_j must be visited in middle of *some* P_i
  - $x_i \in C_j \implies$ can visit c_j during $L \to R$ traversal of P_i. $x_i = 1$ satisfies C_j
  - $\bar{x}_i \in C_j \implies$ can visit c_j during $R \to L$ traveral of P_i. $x_i = 0$ satisfies C_j
- There is a Hamiltonian cycle

  $\iff$ can visit all clause nodes

  $\iff$ there is a truth assignment that satisfies all clauses

Reduction: Details

- n rows (bidirected paths) $P_1, \ldots, P_n$ (one per variable)
- Row has $3m + 3$ vertices, connected to neighbors in forward/backward direction
- First and last vertex of row i connected to first and last of $i + 1$.
- Source s connected to first and last of row 1.
- First and last of row n connected to t.
- Edge (t, s)
- Skeleton has 2^n possible Hamiltonian Cycles, corresponding to truth assignments to $x_1, \ldots, x_n$
  - Traverse P_i L to R $\iff x_i = 1$
  - Traverse P_i R to L $\iff x_i = 0$

Reduction: Clause Gadgets

For each clause C_ℓ construct gadget to restrict possible truth assignments

- New node c_ℓ
- If $x_i \in C_\ell$
  - Add edges $(v_{i,3\ell}, c_\ell)$ and $(c_\ell, v_{i,3\ell+1})$
  - c_ℓ can be visited during L to R traversal of P_i
- If $\neg x_i \in C_\ell$
  - Add edges $(v_{i,3\ell+1}, c_\ell)$ and $(c_\ell, v_{i,3\ell})$
  - c_ℓ can be visited during R to L traversal of P_i

Proof of Correctness

Given a satisfying assignment, construct Hamiltonian Cycle

- If $x_i = 1$ traverse P_i from $L \to R$, else $R \to L$.
- Each C_ℓ is satisfied, so one path P_i is traversed in the correct direction to "splice" c_ℓ into our cycle
- The result is a Hamiltonian Cycle

Given Hamiltonian cycle, construct satisfying assignment:

- If cycle visits c_ℓ from row i, it will also leave to row i because of "buffer" nodes
- Therefore, ignoring clause nodes, cycle traverses each row completely from $L \to R$ or $R \to L$
- Set $x_i = 1$ if P_i traversed $L \to R$, else $x_i = 0$
- Every node c_j visited $\Rightarrow$ every clause C_j is satisfied

Traveling Salesman

TSP. Given n cities and distance function $d(i,j)$, is there a tour that visits all cities with total distance less than D?

Theorem. TSP is NP-Complete

- Clearly in NP.
- Reduction? From $\textsc{Ham-Cycle}$

Clicker

We want to show that $\textsc{Ham-Cycle} \le_P \mathrm{TSP}$. How can we do so?

Given a $\textsc{HamCycle}$ instance $G = (V, E)$ make TSP instance with one city per vertex and...

A. $d(v_i, v_j) = 1$ if $(v_i, v_j) \in E$, else 2. Tour distance: $\le n$?

B. $d(v_i, v_j) = 2$ if $(v_i, v_j) \in E$, else 1. Tour distance: $\le n$?

C. $d(v_i, v_j) = 1$ if $(v_i, v_j) \in E$, else 2. Tour distance: $\le m$?

Reduction from HAM-CYCLE to TSP

Given HAMCYCLE instance $G = (V, E)$ make TSP instance

- One city per vertex
- $d(v_i, v_j) = 1$ if $(v_i, v_j) \in E$, else 2

Claim: there is a tour of distance $\leq n$ if and only if G has a Hamiltonian cycle

- A Hamiltonian cycle clearly gives a tour of length n
- A tour of length n must travel n hops of length 1, which corresponds to a Hamiltonian cycle

HAM-PATH

Similar to Hamiltonian Cycle: is there a *path* that visits every vertex exactly once?

Theorem. HAM-PATH is NP-Complete.

Two proofs:

- Modify 3-SAT to HAM-CYCLE reduction.
- Show that HAM-CYCLE reduces to HAM-PATH

NP-Complete Problems