Dynamic Programming Recipe

- **Step 1:** Devise simple recursive algorithm
  - Flavor: make “first choice”, then recursively solve subproblem
- **Step 2:** Write recurrence for optimal value
- **Step 3:** Design bottom-up iterative algorithm

- Weighted interval scheduling: first-choice is binary
- Rod-cutting: first choice has $n$ options
- Subset Sum: first choice is binary, but need to “add a variable” to recurrence

Subset Sum: Problem Formulation

- **Input**
  - Items 1, 2, ..., $n$
  - Weights $w_i$ for all items (integers)
  - Capacity $W$
- **Goal:** select a subset $S$ whose total weight is as large as possible without exceeding $W$.

Step 1: Recursive Algorithm, Binary Choice

Let $O$ be optimal solution on items 1 through $j$. Is $j \in O$ or not?

```
SubsetSum(j)
if $j = 0$ then return 0
▷ Case 1: $j \notin O$
  $v = SubsetSum(j - 1)$

if $w_j \leq W$ then
  ▷ else skip b/c can’t fit $w_j$
  $v = \max(v, w_j + SubsetSum(j - 1))$
return $v$
```
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SubsetSum($j$)

if $j = 0$ then return 0
$v = $ SubsetSum($j - 1$) ▷ Case 1: $j \notin O$
if $w_j \leq W$ then ▷ Case 2: $j \in O$
$v = \max(v, w_j + $ SubsetSum($j - 1$))
return $v$

Is there a problem in Case 2?
A. No, it is correct.
B. Yes, you need to consider that the $j^{th}$ item may be selected multiple times.
C. Yes, if we take item $j$, the remaining capacity changes.
Second call to SubsetSum($j - 1$) no longer has capacity $W$.
Solution: must add extra parameter (problem dimension)

Step 1: Recursive Algorithm, Add a Variable

Find value of optimal solution $O$ on items $\{1, 2, \ldots, j\}$ when the remaining capacity is $w$

SubsetSum($j, w$)

if $j = 0$ then return 0
$v = $ SubsetSum($j - 1, w$) ▷ Case 1: $j \notin O$
if $w_j \leq w$ then ▷ Case 2: $j \in O$
$v = \max(v, w_j + $ SubsetSum($j - 1, w - w_j$))
return $v$

Step 2: Recurrence

Let $OPT(j, w)$ be the maximum-weight subset of items $\{1, \ldots, j\}$ whose weight does not exceed $w$

$OPT(j, w) = \begin{cases} 
OPT(j - 1, w) & w_j > w \\
\max \left\{ OPT(j - 1, w), w_j + OPT(j - 1, w - w_j) \right\} & w_j \leq w 
\end{cases}$

Base case: $OPT(0, w) = 0$ for all $w = 0, 1, \ldots, W$.

Questions
- Do we need a base case for $OPT(j, 0)$? No
- What is overall optimum to original problem? $OPT(n, W)$

From Recurrence to Iterative (“Turn the Crank”)

$OPT(j, w) = \begin{cases} 
OPT(j - 1, w) & w_j > w \\
\max \left\{ OPT(j - 1, w), w_j + OPT(j - 1, w - w_j) \right\} & w_j \leq w 
\end{cases}$

What size memoization array? $M[j, w]$ for all values of $j$ and $w$ $M[0 \ldots n, 0 \ldots W]$.

What order to fill entries? base case first; RHS before LHS for $j$ from $0 \rightarrow n$, for $w$ from $0 \rightarrow W$.

Which entry stores solution to overall problem? Want $OPT(n, W)$: stored in $M[n, W]$.
Step 3: Iterative Algorithm

SubsetSum(n, W)
Initialize array M[0..n, 0..W]
Set M[0, w] = 0 for w = 0, . . . , W
for j = 1 to n do
    for w = 1 to W do
        if w_j > w then M[j, w] = M[j − 1, w]
        else M[j, w] = max(M[j − 1, w], w_j + M[j − 1, w − w_j])
    return M[n, W]

Running Time? Θ(nW).

Polynomial vs. Pseudo-polynomial

If numbers have m digits, input size is Θ(nm), runtime is Θ(n10^m).

▶ Polynomial time: polynomial in input size (nm)
▶ Pseudo-polynomial: polynomial in number of items (n) and magnitude of numbers (10^m)

For numeric problems, input size is log of magnitude of the numbers. Poly-time algorithm should be polynomial in n and log W.

Subset Sum:
▶ Our solution is pseudo-polynomial
▶ No polynomial algorithm is known

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for j = 1 to n do
    for w = 1 to W do
        if w_j > w then M[j, w] = M[j − 1, w]
        else M[j, w] = max(M[j − 1, w], w_j + M[j − 1, w − w_j])

Suppose we have n items, and the capacity W and weights w_j each have m decimal digits. Then the running time is:
A. Θ(nm)
B. Θ(n log_{10} m)
C. Θ(n10^m)
D. Θ(10^{nm})

Subset Sum

Source: https://www.xkcd.com/287/
Knapsack Problem

Same as subset sum, but now items have value in addition to weight.

Input
- Items $1, 2, \ldots, n$
- Weights $w_j$ for all items (integers)
- Values $v_j$ for all items (integers)
- Capacity $W$

Goal: select subset $S$ whose total value is as large as possible without exceeding $W$. 

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Recall subset-sum recurrence:

$$\text{OPT}(j, w) = \begin{cases} \text{OPT}(j-1, w) & w_j > w \\
\max \{ \text{OPT}(j-1, w), w_j + \text{OPT}(j-1, w-w_j) \} & w_j \leq w \end{cases}$$

How should the blue term be rewritten for the knapsack recurrence?

A. $w_j + \text{OPT}(j-1, w-w_j)$
B. $w_j + \text{OPT}(j-1, w-v_j)$
C. $v_j + \text{OPT}(j-1, w-v_j)$
D. $v_j + \text{OPT}(j-1, w-w_j)$

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Does our knapsack solution still work if the weights and/or values are real numbers instead of integers?

A. It still works if both the values and weights are real numbers.
B. It works if values are real numbers but weights are integers.
C. It works if weights are real numbers but values are integers.
D. It does not work if either the weights or values are real numbers.

Fractional knapsack problem allows partial objects (think grains, sand, fluid). Has simple greedy solution: choose highest value per weight.