COMPSCI 311: Introduction to Algorithms

Lecture 15: Dynamic Programming

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Dynamic Programming Recipe

- Step 1: Devise simple recursive algorithm
  - Flavor: make "first choice", then recursively solve subproblem
- Step 2: Write recurrence for optimal value
- Step 3: Design bottom-up iterative algorithm

- Weighted interval scheduling: first-choice is binary
- Rod-cutting: first choice has n options
- Subset Sum: first choice is binary, but need to "add a variable" to recurrence

Subset Sum: Problem Formulation

- **Input**
  - Items $1, 2, \ldots, n$
  - Weights w_i for all items (integers)
  - Capacity W
- **Goal**: select a subset S whose total weight is as large as possible without exceeding W.
- Subset Sum: need to “add a variable” to recurrence

Step 1: Recursive Algorithm, Binary Choice

Let O be optimal solution on items 1 through j. Is $j \in O$ or not?

SubsetSum(j)

if $j = 0$ **then return** 0

▷ Case 1: $j \notin O$

$v =$ SubsetSum$(j-1)$

▷ Case 2: $j \in O$

if $w_j \leq W$ **then** ▷ else skip b/c can't fit w_j

$v = \max(v, w_j +$ SubsetSum$(j-1)$?$)$

return v

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SubsetSum(j)
  if $j = 0$ **then return** 0
  $v =$ SubsetSum($j - 1$) ▷ Case 1: $j \notin O$
  if $w_j \leq W$ **then** ▷ Case 2: $j \in O$
    $v =$ max(v, $w_j +$ SubsetSum($j - 1$)?)
  return v

Is there a problem in Case 2?

A. No, it is correct.

B. Yes, you need to consider that the j^{th} item may be selected multiple times.

C. Yes, if we take item j, the remaining capacity changes.

Second call to SubsetSum($j - 1$) no longer has capacity W.
Solution: must add extra parameter (problem dimension)

Step 1: Recursive Algorithm, Add a Variable

Find value of optimal solution O on items $\{1, 2, \ldots, j\}$ when the remaining capacity is w

SubsetSum(j,w)

if $j = 0$ **then** return 0

$\triangleright$ Case 1: $j \notin O$

$v =$ SubsetSum($j - 1$, w)

$\triangleright$ Case 2: $j \in O$

if $w_j \leq w$ **then**

$v =$ max(v, $w_j +$ SubsetSum($j - 1$, $w - w_j$))

return v

Step 2: Recurrence

- Let $\text{OPT}(j, w)$ be the maximum weight of any subset of items $\{1, \ldots, j\}$ that does not exceed w

$$\text{OPT}(j, w) = \begin{cases} \text{OPT}(j-1, w) & w_j > w \\ \max \left\{ \begin{array}{c} \text{OPT}(j-1, w) \\ w_j + \text{OPT}(j-1, w - w_j) \end{array} \right\} & w_j \leq w \end{cases}$$

- Base case: $\text{OPT}(0, w) = 0$ for all $w = 0, 1, \ldots, W$.
- Questions
  - Do we need a base case for $\text{OPT}(j, 0)$? No
  - What is overall optimum to original problem? $\text{OPT}(n, W)$

From Recurrence to Iterative ("Turn the Crank")

$$\text{OPT}(j, w) = \begin{cases} \text{OPT}(j-1, w) & w_j > w \\ \max \begin{Bmatrix} \text{OPT}(j-1, w) \\ w_j + \text{OPT}(j-1, w-w_j) \end{Bmatrix} & w_j \le w \end{cases}$$

What size memoization array? $M[j, w]$ for all values of j and w
$M[0 \dots n, 0 \dots W]$

What order to fill entries? base case first; RHS before LHS
for j from $0 \to n$, for w from $0 \to W$

Which entry stores solution to overall problem? Want $\text{OPT}(n, W)$: stored in $M[n, W]$

Step 3: Iterative Algorithm

SubsetSum(n,W)

- Initialize array $M[0..n, 0..W]$
- Set $M[0, w] = 0$ for $w = 0, \ldots, W$
- **for** $j = 1$ to n **do**
  - **for** $w = 1$ to W **do**
    - **if** $w_j > w$ **then** $M[j, w] = M[j-1, w]$
    - **else** $M[j, w] = \max(M[j-1, w], w_j + M[j-1, w-w_j])$
- **return** $M[n, W]$

Running Time? $\Theta(nW)$.

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for $j = 1$ to n **do**
  for $w = 1$ to W **do**
    if $w_j > w$ **then** $M[j, w] = M[j-1, w]$
    else $M[j, w] = \max(M[j-1, w], w_j + M[j-1, w - w_j])$

Suppose we have n items, and the capacity W and weights w_j each have m decimal digits. Then the running time is:

A. $\Theta(nm)$

B. $\Theta(n \log_{10} m)$

C. $\Theta(n10^m)$

D. $\Theta(10^{nm})$

Polynomial vs. Pseudo-polynomial

If numbers have m digits, input size is $\Theta(nm)$, runtime is $\Theta(n10^m)$.

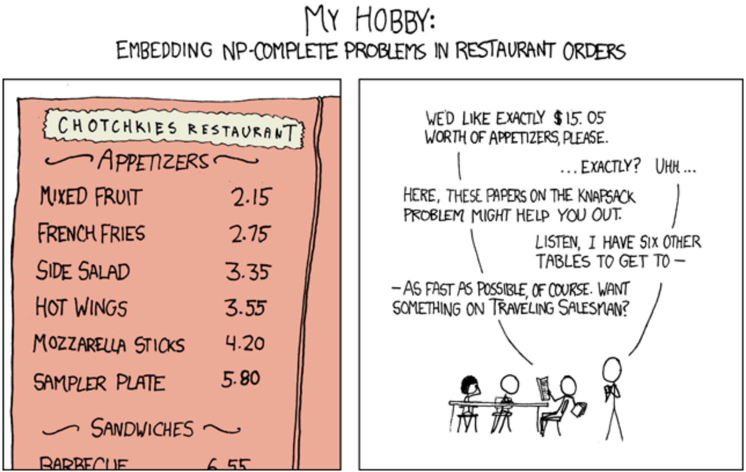
- **Polynomial time**: polynomial in input size (nm)
- **Pseudo-polynomial**: polynomial in number of items (n) and *magnitude* of numbers (10^m)

For numeric problems, input size is *log of magnitude* of the numbers. Poly-time algorithm should be polynomial in n and $\log W$.

Subset Sum:

- Our solution is pseudo-polynomial
- No polynomial algorithm is known

Subset Sum



Source: https://www.xkcd.com/287/

Knapsack Problem

Same as subset sum, but now items have **value** in addition to weight

Input

- Items $1, 2, \ldots, n$
- Weights w_i for all items (integers)
- Values v_i for all items (integers)
- Capacity W

Goal: select subset S whose total **value** is as large as possible without exceeding W.

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Recall subset-sum recurrence:

$$\mathrm{OPT}(j,w) = \begin{cases} \mathrm{OPT}(j-1,w) & w_j > w \\ \max\left\{ \mathrm{OPT}(j-1,w),\ \ w_j + \mathrm{OPT}(j-1, w-w_j) \right\} & w_j \le w \end{cases}$$

How should the blue term be rewritten for the knapsack recurrence?

A. $w_j + \mathrm{OPT}(j-1, w-w_j)$

B. $w_j + \mathrm{OPT}(j-1, w-v_j)$

C. $v_j + \mathrm{OPT}(j-1, w-v_j)$

D. $v_j + \mathrm{OPT}(j-1, w-w_j)$

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Does our knapsack solution still work if the weights and/or values are real numbers instead of integers?

A. It still works if both the values and weights are real numbers.

B. It works if values are real numbers but weights are integers.

C. It works if weights are real numbers but values are integers.

D. It does not work if either the weights or values are real numbers.

Fractional knapsack problem allows partial objects (think: grains, sand, fluid). Has simple **greedy** solution: choose highest value per weight.