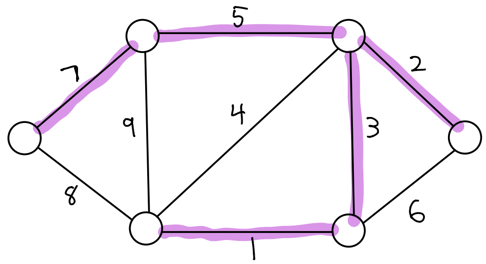
Goals

- Introduce the minimum spanning tree problem

Network Design Problem



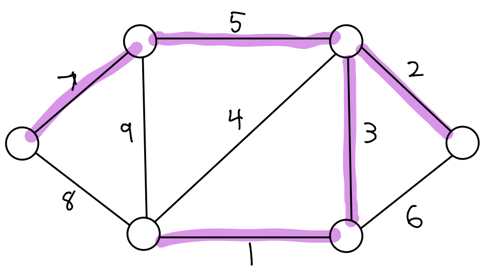
cost(T) =
1+2+3+5+7
= 18

e = (u,v)

c(u,v)

- **Given**: an undirected graph $G = (V, E)$ with edge costs (weights) $c_e > 0$. Assume for now that all edge weights are distinct.
- **Find**: subset of edges $T \subseteq E$ such that (V, T) is connected and the total cost of edges in T is as small as possible

Minimum Spanning Tree Problem



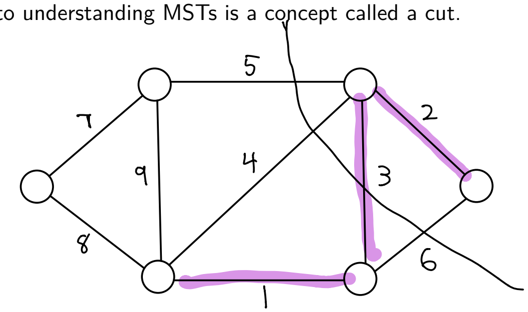
- Call $T \subseteq E$ a *spanning tree* if (V, T) is a tree (*connected*, no cycles)
- **Claim**: in a minimum-cost solution, T is a spanning tree.
- Therefore, we call this the **minimum spanning tree (MST) problem**.

Goals

- State and prove the cut property for minimum spanning trees

Cuts

- A key to understanding MSTs is a concept called a cut.



- **Definition**: A cut is a partition of the nodes into two nonempty subsets $(S, V \setminus S)$.
- **Definition**: Edge $e = (v, w)$ crosses the cut if $v \in S$ and $w \in V \setminus S$.

Cut Property

- **Theorem (cut property)**: Let $e = (v, w)$ be the minimum-weight edge crossing cut $(S, V \setminus S)$ in G.
  Then e belongs to every minimum spanning tree of G.
- Terminology:
  - e is the cheapest or lightest edge across the cut
  - It is safe to add e to a MST
- We will see two different greedy algorithms based on the cut property: Kruskal's algorithm and Prim's algorithm.

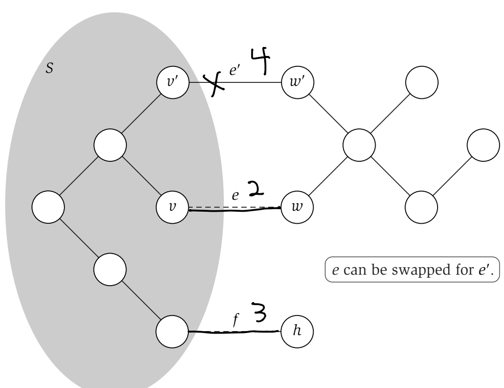
Proof of Cut Property

- Let $e = (v, w)$ be the min-wt edge across cut $(S, V \setminus S)$ and suppose for contradiction that T is MST but does not include e

$T \in$ MST

G ------

but not T



Proof of Cut Property

- Let $e = (v, w)$ be the min-wt edge across cut $(S, V \setminus S)$ and suppose for contradiction that T is MST but does not include e
- There is a path from v to w in T
- Let $e' = (v', w')$ be the first edge on this path that crosses the cut
- Let $T' = T + \{e\} - \{e'\}$
- T' is still a spanning tree:
  - Connected: any path in T that needed e' can be routed via e instead
  - No cycles: adding e creates one cycle, removing e' destroys it
- But since e is the lightest edge from S to $V \setminus S$,

$$w(T') = w(T) - w(e') + w(e) < w(T)$$

$w(T) - 4 + 2$

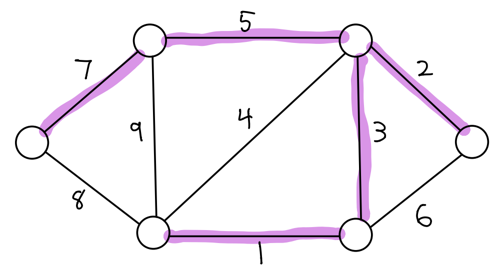
$\Rightarrow\Leftarrow$ T is not MST

Goals

- Use cut property to derive Kruskal's algorithm and prove it is correct

Kruskal's algorithm

- Armed with the cut property, how can we find a MST?



- **Kruskal's algorithm**: add edges in order of weight as long it does not create a cycle

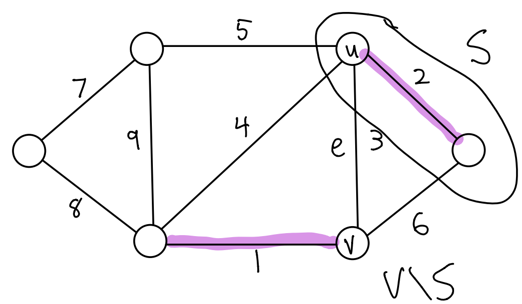
Kruskal's algorithm

Assume edges are numbered $e = 1, \dots, m$
Sort edges by weight so $c_1 \leq c_2 \leq \dots \leq c_m$
Initialize $T = \{\}$
for $e = 1$ to m **do**
  if adding e to T does not form a cycle **then**
    $T = T \cup \{e\}$

Exercise: argue correctness (use cut property)

Kruskal's algorithm proof

- Let T be partial spanning tree just before adding $e = (u, v)$



- What cut can we use to prove that e belongs to MST?

Kruskal's algorithm proof

- Let T be partial spanning tree just before adding $e = (u, v)$
- Let S be the connected component of T that contains u
- e crosses $(S, V \setminus S)$, otherwise adding e would create cycle
- No other edge crossing $(S, V \setminus S)$ has been considered yet; it could have been added without creating a cycle
- $\implies e$ is the cheapest edge across $(S, V \setminus S)$
- $\implies e$ belongs to every MST (cut property)

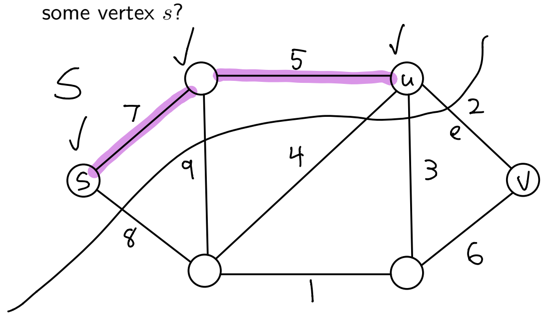
- Every edge added belongs to the MST
- By design, the algorithm creates no cycles and doesn't stop until (V, T) is connected
- $\implies T$ is MST

Goals

- Use cut property to derive Prim's algorithm and prove it is correct

Prim's Algorithm

- What if we want to grow a tree as a *single* connected component starting from some vertex s?



7, 5, 2, 3, 1

- **Prim's algorithm**: let S be the component containing s. Add cheapest edge from S to V\S.

Prim's Algorithm

Initialize $T = \{\}$
Initialize $S = \{s\}$
while $|S| < n$ **do**
  Let $e = (u, v)$ be the minimum-cost edge from S to $V - S$
  $T = T \cup \{e\}$
  $S = S \cup \{v\}$

Correctness? use cut property

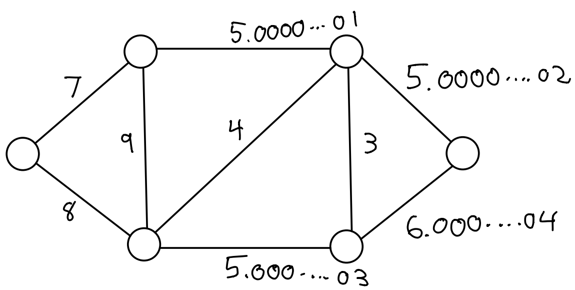
Prim's algorithm proof

- Let T be the partial spanning tree just before adding edge e
  - Let S be the connected component containing s
  - By construction, e is the cheapest edge across the cut $(S, V - S)$
  - Therefore, e belongs to every MST
- So, every edge added belongs to the MST.
- The algorithm creates no cycles and does not stop until the graph is connected, therefore, the final output is a spanning tree.
- The final output is a minimum-spanning tree.

Goals

- Remove distinctness assumption
- Give implementation of Prim's algorithm and analyze its running-time

Remove Distinctness Assumption?



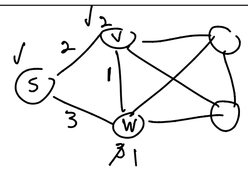
- Hack: break ties by perturbing each edge weight by a tiny unique amount.
- Implementation: break ties in an arbitrary but consistent way (e.g., lexicographic)
- (There is a more "elegant" way that requires a stronger cut property.)

Implementation of Prim's algorithm

Initialize $T = \{\}$
Initialize $S = \{s\}$
while T is not a spanning tree **do**
  Let $e = (u, v)$ be the minimum-cost edge from S to $V - S$
  $T = T \cup \{e\}$
  $S = S \cup \{\cancel{e}\}$ v   mark v "attached"

What does this remind you of?

Prim Implementation



Set $A = V$ ▷ Unattached nodes
Set $a(v) = \infty$ for all nodes ▷ Attachment cost
Set $a(s) = 0$
Set edgeTo(s) = null ▷ Attachment edge
while A not empty **do** ▷ Nodes left to attach
  Extract node $v \in A$ with smallest $a(v)$ value
  Set $T = T \cup$ edgeTo(v)
  for all edges (v, w) where $w \in A$ **do**
    if $c(v, w) < a(w)$ **then** ▷ Cheaper edge to w?
      $a(w) = c(v, w)$
      edgeTo(w) = (v, w)

assume $m \geq n-1$

Nearly identical to Dijkstra. Priority queue for $A \to O(m \log n)$

Dijkstra $O((m+n)\log n)$

Goals

- Describe implementation of Kruskal's algorithm

Kruskal Implementation?

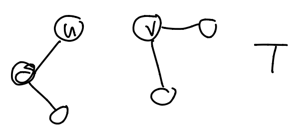
Sort edges by weight so $c_1 \leq c_2 \leq \ldots \leq c_m$
Initialize $T = \{\}$
for $e = 1$ to m **do**
  if adding $e = (u, v)$ to T does not form a cycle **then**
    $T = T \cup \{e\}$

$O(\underbrace{\#\text{edges of } T}_{\leq n} + \underbrace{\#\text{nodes of } T}_{n})$

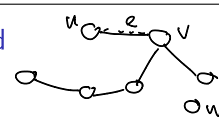
$O(n)$

Ideas? use BFS

$O(mn)$



Kruskal Implementation: Union-Find



Idea: use clever data structure to maintain connected components of growing spanning tree. Should support:

- find(v): return name of set containing v
- Union(A, B): merge two sets

for $e = 1$ to m **do**
  Let u and v be endpoints of e $O(1)$
  if find(u) != find(v) **then** $O(\log n)$ ▷ Not in same component?
    $T = T \cup \{e\}$ $O(1)$
    Union(find(u), find(v)) $O(1)$ ▷ Merge components

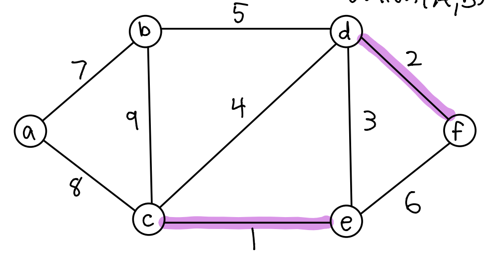
Goal: union = $O(1)$, find = $O(\log n) \implies O(m \log n)$ overall

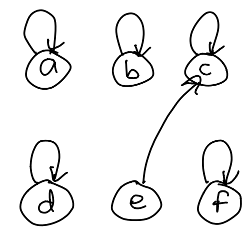
Goals

- Describe union-find data structure

Union-Find Data Structure

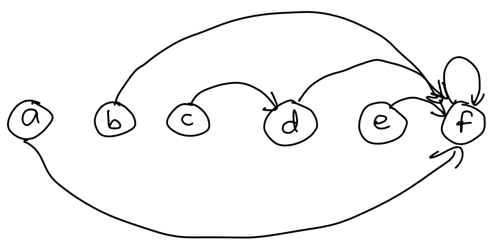
find(v)
Union(A,B)





- Each set elects a representative to act as the "name" of the set
- Nodes point to their representative
- Initially, all nodes point to themself

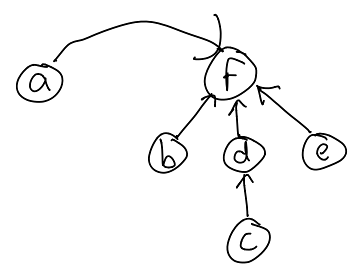
Union-Find Data Structure



- Union(e, f)
- Union(c, d)
- Union(d, f)
- Union(b, f)
- Union(a, f)

- Time for union? $O(1)$

Union-Find Data Structure



find(c) = f

- Union(a, f): which pointer should be updated?
- **Convention**: smaller set updates its pointer
- Time for find? proportional to depth of tree

Union-Find Data Structure

Claim: let d = depth and k = # nodes in set. Then $d \le \log_2(k)$.

- $\implies$ find is $O(\log n)$

Equivalent claim: $k \ge 2^d$

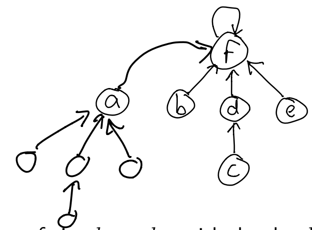
Proof: by induction.

Base case: $d = 0, k = 1$ ✓

Induction Step

$k_L \ge 2^{d_L}$

$k_R \ge 2^{d_R}$



$d = d_L + 1$

$k \ge 2 \cdot k_L$

- Consider union of sets of size $k_L < k_R$ with depths d_L and d_R
- Case 1: depth same as right subtree; size of set increases
  - $d = d_R$
  - $k = k_L + k_R \ge k_R \ge 2^{d_R} = 2^d$
- Case 2: depth one more than left subtree; size of set at least doubles
  - $d = d_L + 1$
  - $k = k_L + k_R \ge 2k_L \ge 2 \cdot 2^{d_L} = 2^{d_L+1} = 2^d$
- In both cases, $k \ge 2^d$ ✓

$d \le \log_2(k) \Rightarrow$ find is $O(\log(n))$

Union-Find Wrap-Up

- Union is $O(1)$: update one pointer
- Find is $O(\log n)$: follow at most $\log_2(n)$ pointers to find representative of set
- m union/find operations takes $O(m \log n)$ time
- Better: path compression $\Rightarrow$ find in nearly constant time