CMPSCI 311: Introduction to Algorithms

Shortest Paths

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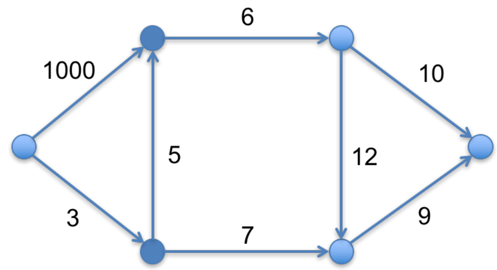
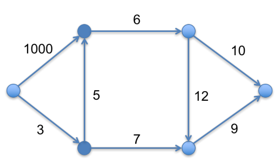
Shortest Paths Problem

Problem: find shortest paths in a directed graph with edge *lengths* (the Google maps problem)



Let's Formalize the Problem



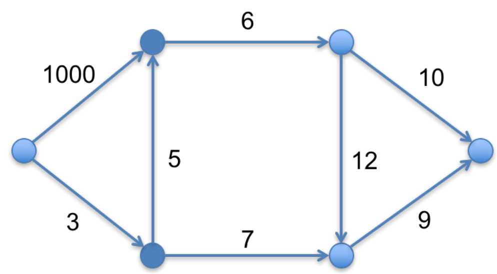
- Directed graph $G = (V, E)$ with edge *lengths* $\ell(e) > 0$
- Define *length* of path P consisting of edges $e_1, e_2, \ldots, e_k$ as

$$\ell(P) = \ell(e_1) + \ell(e_2) + \ldots + \ell(e_k)$$

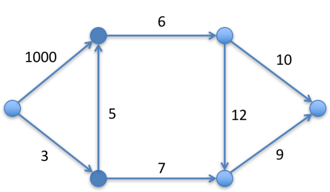
- Starting node s
- Let $d(v)$ be the length of shortest $s \rightsquigarrow v$ path.
- **Problem:** Can we efficiently find $d(v)$ for all nodes $v \in V$?

Shortest Paths Problem

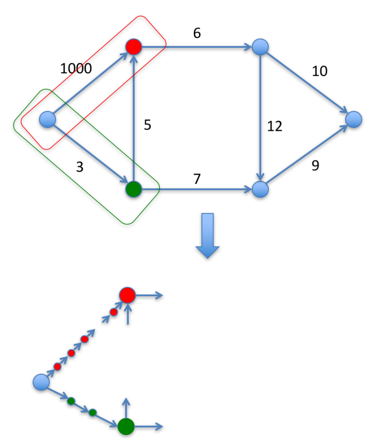
Suppose all edges have integer length. Can we use BFS to solve this problem?



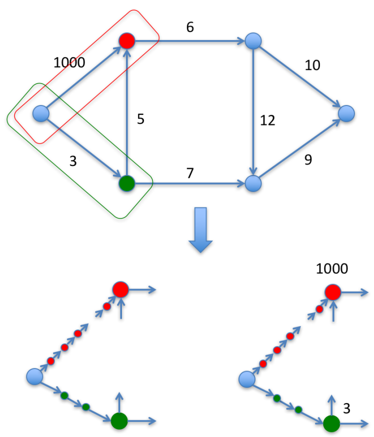
Shortest Paths Problem



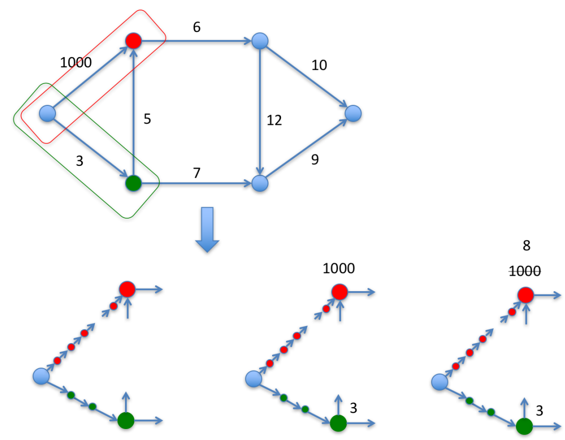
Shortest Paths Problem



Shortest Paths Problem



Shortest Paths Problem



Shortest Paths Problem

Idea: keep track of the "wavefront"

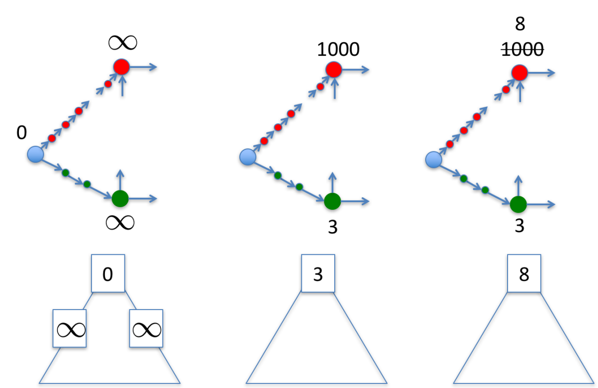
- $d'(v)$ — best tentative arrival time so far for node v
- $d(v)$ — actual arrival time

What's required to keep track of the wavefront?

- Find next arrival: find node v with smallest $d'(v)$
- Set arrival time: $d(v) = d'(v)$
- Update $d'(v)$ for neighbors of v if they get better "offers"

What data stucture supports find smallest and update values?
Priority queue.

Shortest Paths Problem



Dijkstra's Algorithm

Set $A = V$ ▷ Priority queue
Set $d'(v) = \infty$ for all nodes ▷ Tentative arrival time
Set $d'(s) = 0$
while A not empty **do** ▷ Nodes left to explore
  Extract node $v \in A$ with smallest $d'(v)$ value
  Set $d(v) = d'(v)$ ▷ Wave arrives at v
  for all edges (v, w) where $w \in A$ **do**
    if $d(v) + \ell(v, w) < d'(w)$ **then** ▷ Better offer?
      $d'(w) = d(v) + \ell(v, w)$
    end if
  end for
end while

Running Time?

Use heap-based priority queue for A

Set $A = V$
Set $d'(v) = \infty$ for all nodes
Set $d'(s) = 0$
while A not empty **do**
  Extract node $v \in A$ with smallest $d'(v)$ value ▷ Extract-min
  Set $d(v) = d'(v)$
  for all edges (v, w) where $w \in A$ **do**
    if $d(v) + \ell(v, w) < d'(w)$ **then**
      $d'(w) = d(v) + \ell(v, w)$ ▷ Update-key
    end if
  end for
end while

- n extract-min operations. $O(n \log n)$
- m update-key operations. $O(m \log n)$
- **Total**: $O((m + n) \log n)$

Finding the Actual Path

Keep track of prev(v) = node that last updated arrival time $d'(v)$ = predecessor in shortest path

Set $A = V$
Set $d'(v) = \infty$ for all nodes
Set prev(v) = null
Set $d'(s) = 0$
while A not empty **do**
  Extract node $v \in A$ with smallest $d'(v)$ value
  Set $d(v) = d'(v)$
  for all edges (v, w) where $w \in A$ **do**
    if $d(v) + \ell(v, w) < d'(w)$ **then**
      $d'(w) = d(v) + \ell(v, w)$
      prev(w) = v
    end if
  end for
end while

Proof of Correctness

- Let $S = V \setminus A$ be the set of *explored* nodes at any point in the algorithm—those v for which we have assigned $d(v)$
- **Observation**: for $v \notin S$, the value $d'(v)$ is the minimum value $d(u) + \ell(u, v)$ over all edges (u, v) where $u \in S$, $v \notin S$.
  - Picture
  - Interpretation: length of shortest path to v that remains in S until final hop.
- **Claim (invariant)**: for $v \in S$, the value $d(v)$ is the length of the shorest $s \rightsquigarrow v$-path
- Proof on board. By induction on $|S|$

Proof (by induction)

- *Base case:* Initially $S = \{s\}$ and $d(s) = 0$. ✓
- *Induction step:*
  - Assume the invariant is true after the kth execution of the while loop, when $|S| = k$.
  - Let v be the next node added to S, and let (u, v) be the preceding edge. Then $d'(u) = d(u) + d(u, v)$, and $d'(u) \leq d'(x)$ or any node x.
  - Let P_u be the shortest $s \rightsquigarrow u$ path, which has length $d(u)$
  - Let $P_v = P_u \cup (u, v)$ be the path found by Dijkstra, which has length $\ell(P_v) = d'(v) = d(u) + \ell(u, v)$
  - Consider any other $s \rightsquigarrow v$ path P. We'll argue that P is *already* longer than P_v by the time it first leaves S.
  - Let (x, y) be the first edge in P with $x \in S, y \notin S$, and let P' be the subpath of P from s to x
  - Then,

$$\ell(P) \geq \ell(P') + \ell(x, y) \geq d(x) + \ell(x, y) \geq d'(y) \geq d'(v) = \ell(P_v)$$