

## COMPSCI 311: Introduction to Algorithms

### Lecture 8: Greedy Algorithms – Exchange Arguments

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## Algorithm Design—Greedy

Greedy: make a single “greedy” choice at a time, don’t look back.

	Greedy
Formulate problem	
Design algorithm	
Prove correctness	✓
Analyze running time	
Specific algorithms	Dijkstra, MST

Focus is on proof techniques

- ▶ Last time: “greedy stays ahead” (inductive proof)
- ▶ This time: exchange argument

## Scheduling to Minimize Lateness

- ▶ You have a very busy month:  $n$  assignments are due, with different deadlines

Assignments:

1: |---| (len=1, due=2)  
2: |---o---| (len=2, due=5)  
3: |---o---o---| (len=3, due=6)  
4: |---o---| (len=2, due=7)

Deadlines:

          d1          d2 d3 d4  
|---|---|---|---|---|---|---|---|  
0 1 2 3 4 5 6 7 8 9

- ▶ How should you schedule your time to “minimize lateness”?

## Scheduling to Minimize Lateness

Let’s formalize the problem. The input is:

- ▶  $t_j$  = length (in days) to complete assignment  $j$  (or “job”  $j$ )
- ▶  $d_j$  = deadline for assignment  $j$

What does a schedule look like?

- ▶  $s_j$  = start time for assignment  $j$  (selected by algorithm)
- ▶  $f_j = s_j + t_j$  finish time

How to evaluate a schedule?

- ▶ Lateness of assignment  $j$  is  $\ell_j = \begin{cases} 0 & \text{if } f_j \leq d_j \\ f_j - d_j & \text{if } f_j > d_j \end{cases}$
- ▶ Maximum lateness  $L = \max_j \ell_j$

**Goal:** schedule so maximum lateness is as small as possible

## Clicker

True or false: an algorithm to minimize maximum lateness will also find a schedule that is not late, if one exists.

- A. True
- B. False, because the lateness function is not linear
- C. False, because it minimizes the maximum lateness, whereas we want all jobs to have lateness zero

## Possible Greedy Approaches

- ▶ **Note:** scheduling work back-to-back (no idle time) can't hurt  
⇒ schedule determined just by order of assignments

1: |---| (len=1, due=2)  
2: |---o---| (len=2, due=5)  
3: |---o---o---| (len=3, due=6)  
4: |---o---| (len=2, due=7)

- ▶ What order should we choose?
  - ▶ *Shortest Length:* ascending order of  $t_j$ .
  - ▶ *Smallest Slack:* ascending order of  $d_j - t_j$ .
  - ▶ *Earliest Deadline:* ascending order of  $d_j$ .

## Clicker

Suppose we have two jobs with lengths 1 and 10. Which idea below can we use to set deadlines to “break” smallest slack time so it does not find an optimal ordering?

- A. Set the deadlines so both jobs have slack zero.
- B. Set the deadlines so the short job has a little slack, and the long job has none.
- C. Set the deadlines so the long job has a little slack, and the short job has none.

## Proposed Algorithm

So far, only **earliest deadline first** is optimal in all the examples we've tried.

Next, we'll prove that it's always optimal.

## Clicker

If two jobs have the same deadline, the earliest deadline first algorithm should schedule:

- A. The shortest job first, because that has a higher chance of finishing before the deadline
- B. The longest job first, because then its lateness will be minimized
- C. Does not matter

## Identical Maximum Lateness

**Claim:** If in an EDF schedule, we swap two jobs with the same deadline, we get the same maximum lateness.

**Proof:** Since the schedules are EDF, all jobs with the same deadline are scheduled in a consecutive block.

Among those, the last one has the maximum lateness.

That finishing time does not change by swapping schedules within the block.

**Corollary** All EDF schedules have the same maximum lateness.

## Exchange Argument (False Start)

Assume jobs ordered by deadline  $d_1 \leq d_2 \leq \dots \leq d_n$ , so the greedy ordering is simply  $A = 1, 2, \dots, n$ . **Claim:**  $A$  is optimal

**Proof attempt:** Suppose for contradiction that  $A$  is not optimal. Then, there is an optimal solution  $O \neq A$ .

- 1. Since  $O \neq A$ , there must be two jobs  $i$  and  $j$  that are out of order in  $O$ .
- 2. Suppose we could show that swapping the jobs  $i$  and  $j$  that are out of order gives a *better* solution  $O'$ .
- 3. This would mean  $O$  is *not* optimal, a contradiction. Therefore,  $A$  must be optimal.

**Problem?** We can't show 2. It's true that  $O'$  is *no worse* than  $O$ , but this means  $O'$  may still be optimal. [Example?](#)

## Exchange Argument (Correct)

Suppose  $O$  optimal and  $O \neq A$ . Then we can modify  $O$  to get a new solution  $O'$  that is:

- 1. No worse than  $O$
- 2. Closer to  $A$  in some measurable way

$$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \dots \rightarrow A(\text{optimal})$$

**High-level idea:** gradually transform an arbitrary optimal solution  $O$  into  $A$  without hurting solution, thus preserving optimality. Conclude that  $A$  is optimal.

**Concretely:** show 1 and 2 above.

## Exchange Argument for Scheduling to Minimize Lateness

Recall  $A = 1, 2, \dots, n$ . For  $O \neq A$ , say there is an **inversion** if  $i$  comes before  $j$  but  $j < i$  (thus  $d_j \leq d_i$ )

**Claim:** if  $O$  has an inversion,  $O$  has a **consecutive inversion**—one where  $i$  comes immediately before  $j$ . Why?

**Main result:** let  $O \neq A$  be an optimal schedule. Then  $O$  has a consecutive inversion  $i, j$ . We can swap  $i$  and  $j$  to get a new schedule  $O'$  such that:

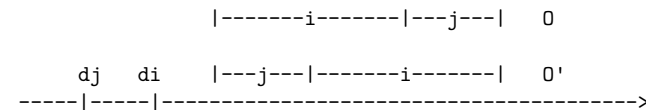
1.  $O'$  has one less inversion than  $O$
2. Maximum lateness of  $O'$  is at most maximum lateness of  $O$

**Proof:**

1. Obvious
2. Next slide(s)

## Proof (Lateness does not increase)

Swapping a consecutive inversion ( $i$  precedes  $j$ ;  $d_j \leq d_i$ )



Consider the lateness  $\ell'_k$  of each job  $k$  in  $O'$ :

- ▶ If  $k \notin \{i, j\}$ , then lateness is unchanged:  $\ell'_k = \ell_k$
- ▶ Job  $j$  finishes earlier in  $O'$  than  $O$ :  $\ell'_j \leq \ell_j$
- ▶ Finish time of  $i$  in  $O' =$  finish time of  $j$  in  $O$ . Therefore

$$\ell'_i = f'_i - d_i = f_j - d_i \leq f_j - d_j = \ell_j$$

**Conclusion:**  $\max_k \ell'_k \leq \max_k \ell_k$ . Therefore  $O'$  is still optimal.

## Wrap-Up (Exchange Argument)

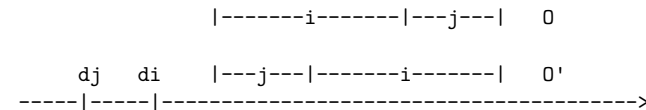
For any optimal  $O \neq A$  we showed that we could transform  $O$  to  $O'$  such that:

1.  $O'$  is still optimal
2.  $O'$  has one less inversion than  $O$

$$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \dots \rightarrow A(\text{optimal})$$

Since there are at most  $\binom{n}{2}$  inversions, by repeating the process a finite number of times we see that  $A$  is optimal.

## Clicker



Consider the *total* lateness  $\ell'_i + \ell'_j$  in the new schedule. Which fact about total lateness follows from our argument?

- A. It is no more than  $2\ell_i$
- B. It is no more than  $\ell_i + \ell_j$
- C. It is no more than  $2\ell_j$
- D. None of the above

B would imply EDF is also optimal for minimizing total lateness. It is not. There is no known polynomial time algorithm for minimizing total lateness.

## Wrap-Up: Greedy Algorithms

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Proof techniques

- ▶ Last time: “greedy stays ahead” (inductive proof) ✓
- ▶ This time: exchange argument ✓