COMPSCI 311 Introduction to Algorithms

Lecture 6: Bipartite Testing, Directed Graphs, Topological Ordering

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Review and Outlook

- Graph traversal by BFS/DFS
  - Different versions of general exploration strategy
  - $O(m + n)$ time
  - Produce trees with useful properties (for other problems)
  - Basic algorithmic primitive — used in many other algorithms (path from s to t, connected components)
- Bipartite testing
- Directed graphs
  - Traversal
  - Topological sorting
  - (Strong connectivity)

Bipartite Graphs

Definition Graph $G = (V, E)$ is bipartite if V can be partitioned into sets X, Y such that every edge has one end in X and one in Y.

Can color nodes red/blue s.t. no edges between nodes of same color.

Examples

- Bipartite: student-college graph in stable matching
- Bipartite: client-server connections
- Not bipartite: "odd cycle" (cycle with odd $\#$ of nodes)

Claim (easy): If G contains an odd cycle, it is not bipartite.

Bipartite Testing

Question Given $G = (V, E)$, is G bipartite?

Algorithm? **Idea**: run BFS from any node s

- $L_0 =$ red
- $L_1 =$ blue
- $L_2 =$ red
- ...
- Even layers red, odd layers blue

What could go wrong? Edge between two nodes at same layer.

Algorithm

Run BFS from any node s

if there is an edge between two nodes in same layer **then**

  Output "not bipartite"

else

  Output "bipartite" with $X =$ even layers and $Y =$ odd layers

Correctness

Remember the **fact about BFS**: every edge connects nodes in the same layer or in adjacent layers (i.e., one even, one odd).

Proof structure:

1. If the algorithms outputs "bipartite", then *all* edges connect nodes in an even layer (X) and an odd layer (Y), so G is bipartite. $\checkmark$
2. If the algorithm outputs "not bipartite", then there is an edge between two nodes in the same layer. We will show this implies that G has an odd cycle, so G is not bipartite.

Proof

Claim: if there is an edge between two nodes in the same layer, then G has an odd cycle.

- Let T be BFS tree of G and suppose (x, y) is an edge between two nodes in the layer j
- Let $z \in L_i$ be the least common ancestor of x and y
  (Useful in proofs: take least/greatest item with some property)
  - Let P_{zx} = path from z to x in T
  - Let P_{yz} = path from z to y in T
  - The path that follows P_{zx} then edge (x, y) then P_{yz} is a cycle of length $2(j - i) + 1$, which is odd
- The claim is proved, which completes the proof of the algorithm

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Which of the following is true?

A. If G is bipartite, then G does not have an odd cycle
B. If G does not have an odd cycle, then G is bipartite
C. Both A and B
D. Neither A nor B

Directed Graphs

$G = (V, E)$

- $(u, v) \in E$ is a *directed* edge
- u points to v
- $e = (u, v)$ leaves u, enters v, is an outgoing edge from u, incoming edge to v

Examples

- Facebook / LinkedIn: undirected
- Twitter/X/ Bluesky / Instagram: directed
- Web: directed
- Road network: directed (discuss)

Directed Graph Definitions

Most definitions extend naturally to directed graphs by mapping the word "edge" to "directed edge"

- **Directed path**: sequence $P = v_1, v_2, \ldots, v_{k-1}, v_k$ such that each consecutive pair v_i, v_{i+1} is joined by a *directed edge* in G. A $v_1 \to v_k$ path.
- **Directed cycle**: directed path with $v_1 = v_k$
- When referring to a directed graph, the words "path" and "cycle" mean "directed path" and "directed cycle"
- **Connected**? **Connected component**? More subtle, because now there can be a path from s to t but not vice versa. More later.

Directed Graph Traversal

Reachability. Find all nodes *reachable* from some node s. All nodes v with $s \to v$ path.

$s \to t$ shortest path. What is the length of the shortest directed path from s to t?

Algorithm?

Directed Graph Traversal

BFS/DFS naturally extend to directed graphs.

BFS(s):

```
mark s as "discovered"
L[0] ← {s}, i ← 0
while L[i] is not empty do
    L[i+1] ← empty list
    for all nodes v in L[i] do
        for all edges (v, w) leaving v do
            if w is not marked "discovered" then
                mark w as "discovered"
                put w in L[i+1]
    i ← i+1
```

Directed Graph Traversal

Find all nodes v with $v \to t$ path? BFS following edges in reverse direction.

Useful to keep adjacency lists for both outgoing and incoming edges.

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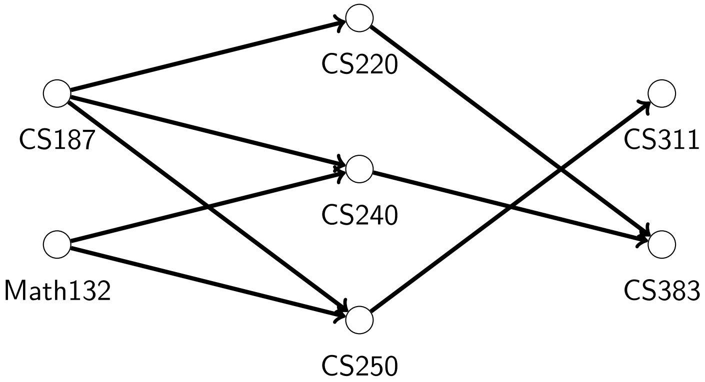
Suppose G is a directed path on n vertices and BFS is called repeatedly starting from any unexplored vertex until all nodes are explored. What is the maximum number of times BFS may be called?

A. $n - 1$
B. n
C. 1
D. m

Directed Acyclic Graphs

Definition: A directed acyclic graph (DAG) is a directed graph with no cycles.

Models *dependencies*, e.g. course prerequisites:



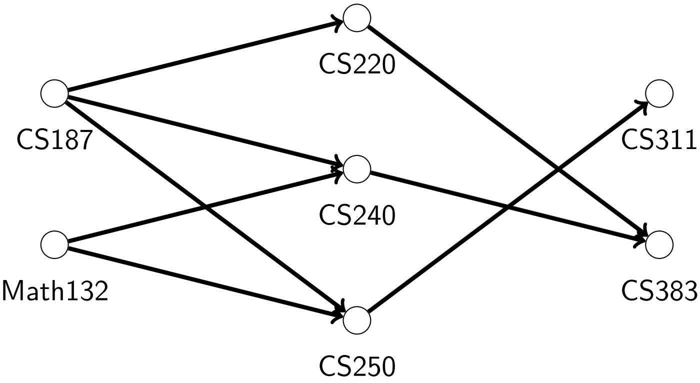
Topological Sorting

Definition: A topological ordering of a directed graph is an ordering of the nodes such that all edges go "forward" in the ordering

- A way to order the classes so all prerequisites are satisfied
- Label nodes $v_1, v_2, \ldots, v_n$ such that
- For all edges (v_i, v_j) we have $i < j$

Q: Is a topological ordering possible for any directed graph?

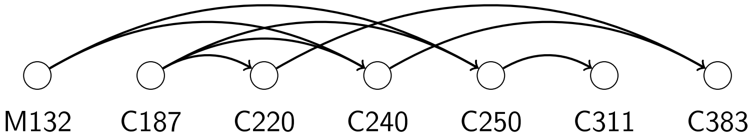
Topological Sorting



Exercise

1. Find a topological ordering.
2. Devise an algorithm to find a topological ordering.

Topological Ordering



Claim If G has a topological ordering, then G is a DAG.

Topological Sorting

Problem Given DAG G, compute a topological ordering for G.

topo-sort(G)

    while there are nodes remaining **do**
        Find a node v with no incoming edges
        Place v next in the order
        Delete v and all of its outgoing edges from G

Running time? Can show it's $O(m + n)$

Topological Sorting Analysis

What to prove:

1. Algorithm can always find an ordering
2. The ordering is a topological ordering.

Sketch of analysis:

- In a DAG G, there is always a node v with no incoming edges. Try to prove.
- Any such node v can be first in the topological ordering.
- Removing v from G produces a new DAG G'.
- The node v followed by a topological ordering for G' is a topological ordering for G.

DAGs and Topological Orderings

Theorem: G is a DAG if and only if G has a topological ordering.

Proof:

1. If G is a DAG, the algorithm finds a topological ordering.
2. If G is not a DAG then G does not have a topological ordering.

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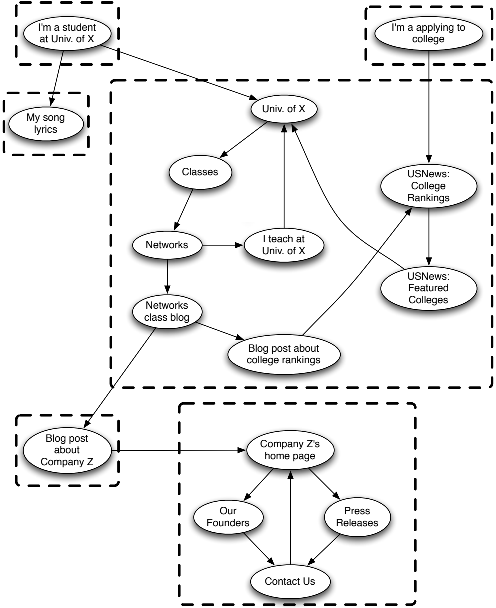
The maximum number of edges in a DAG with n nodes is

A. $2(n-1)$
B. $2n-1$
C. $n(n-1)/2$
D. $n(n-1)$

Directed Graph Connectivity



Strongly connected graph: graph with directed path between any pair of nodes.

Strongly connected component (SCC): maximal subset of nodes with directed path between any pair.

SCCs can be found in time $O(m+n)$. (Tarjan, 1972)

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Consider the graph G' whose nodes are SCCs and there is an edge from C to D if any node in C has an edge to D. Which of the following is always true?

A. G' is strongly connected
B. G' has a cycle
C. G' has at least $n/2$ nodes
D. G' is a DAG