COMPSCI 311 Introduction to Algorithms

Lecture 5: Graph Traversal

Dan Sheldon

University of Massachusetts Amherst

Graph Traversal

Thought experiment. World social graph.

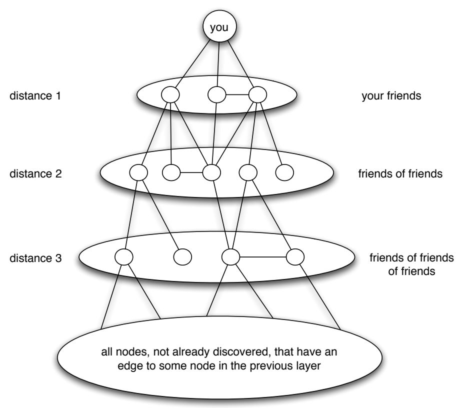
- Is it connected?
- If not, how big is largest connected component?
- Is there a path between you and King Charles III?

How can you tell algorithmically?

Answer: graph traversal! (BFS/DFS)

Breadth-First Search

Explore outward from starting node s by distance. "Expanding wave"



Breadth-First Search: Layers

Explore outward from starting node s.

Define layer L_i = all nodes at distance exactly i from s.

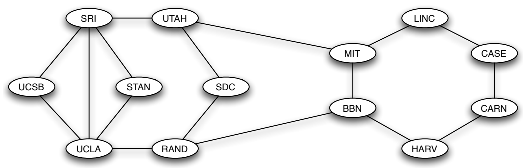
Layers

- $L_0 = \{s\}$
- L_1 = nodes with edge to L_0
- L_2 = nodes with an edge to L_1 that don't belong to L_0 or L_1
- ...
- L_{i+1} = nodes with an edge to L_i that don't belong to any earlier layer.

Observation: There is a path from s to t if and only if t appears in some layer.

Clicker

How many nodes are in layer 2, starting a BFS from MIT ?



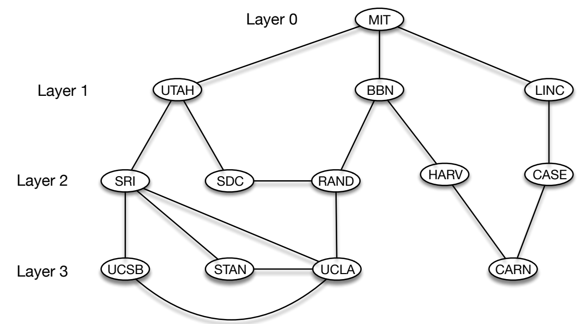
A. 4

B. 5

C. 6

D. None of the above

BFS Layers



BFS Implementation

BFS(s):
  mark s as "discovered"
  $L[0] \leftarrow \{s\}$, $i \leftarrow 0$ ▷ Discover s
  while $L[i]$ is not empty **do**
    $L[i+1] \leftarrow$ empty list
    for all nodes v in $L[i]$ **do**
      for all neighbors w of v **do** ▷ Explore v
        if w is not marked "discovered" **then**
          mark w as "discovered" ▷ Discover w
          put w in $L[i+1]$
    $i \leftarrow i+1$

Running time? How many times does each line execute?
(For now, assume graph is connected)

BFS Running Time

BFS(s):
  mark s as "discovered" ▷ 1
  $L[0] \leftarrow \{s\}$, $i \leftarrow 0$ ▷ 1
  while $L[i]$ is not empty **do** ▷ $\leq n$
    $L[i+1] \leftarrow$ empty list ▷ $\leq n$
    for all nodes v in $L[i]$ **do** ▷ n
      for all neighbors w of v **do** ▷ $2m$
        if w is not marked "discovered" **then** ▷ $2m$
          mark w as "discovered" ▷ n
          put w in $L[i+1]$ ▷ n
    $i \leftarrow i+1$ ▷ $\leq n$

Running time: $\Theta(m+n)$

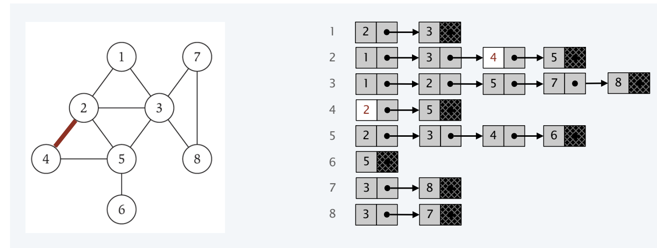
BFS Running Time

BFS running time: $\Theta(m+n)$

- Another way to think about it: "touch each node and edge" a constant number of times
- Hidden assumption: can iterate over neighbors of v efficiently. . .

Graph Representation: Adjacency Lists

Each node keeps **list of neighbors**



- Each edge stored twice
- Space? $\Theta(m+n)$
- Time to check if (u,v) is an edge? $O(\text{degree}(u))$
  (degree = number of neighbors)
- Time to iterate over all neighbors of v? $O(\text{degree}(u))$

Clicker

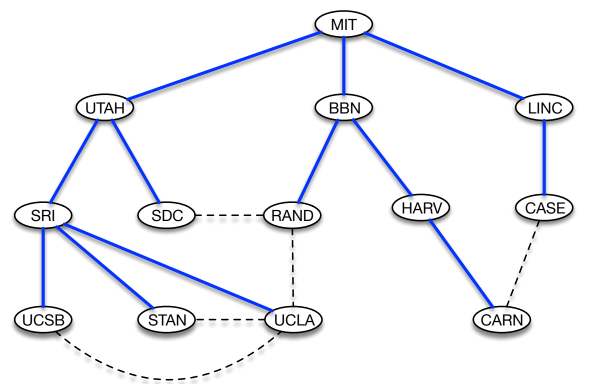
Let $q = \sum_{v\in V} \text{degree}(v)$ (this is the sum of degrees of all nodes in the graph)

Which one of the following is false?

A. q is twice the number of edges
B. q is n times the average degree
C. q is $\Theta(m+n)$ if $m \geq n$
D. None of the above

BFS Tree

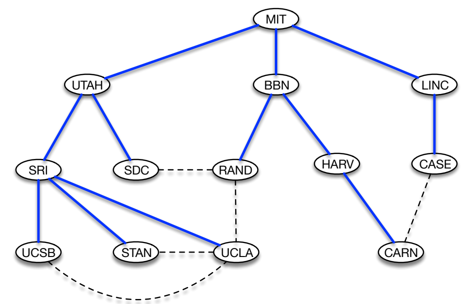
We can use BFS to make a tree. (blue: "tree edges", dashed: "non-tree edges")



BFS Tree

BFS(s):
  mark s as "discovered"
  $L[0] \leftarrow \{s\}$, $i \leftarrow 0$
  $T \leftarrow$ empty
  while $L[i]$ is not empty **do**
    $L[i+1] \leftarrow$ empty list
    for all nodes v in $L[i]$ **do**
      for all neighbors w of v **do**
        if w is not marked "discovered" **then**
          mark w as "discovered"
          put w in $L[i+1]$
          put (v, w) in T
    $i \leftarrow i + 1$

BFS Tree



Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G. Then the layer of x and y in T differ by at most 1.

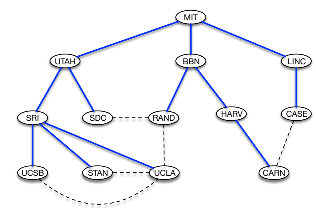
BFS and non-tree edges

Claim: let T be the tree discovered by BFS on graph $G = (V, E)$, and let (x, y) be any edge of G. Then the layer of x and y in T differ by at most 1.

Proof

- Let (x, y) be an edge
- Assume x is discovered first and placed in L_i
- Then $y \in L_j$ for $j \geq i$
- When neighbors of x are explored, y is either already in L_i or L_{i+1}, or is discovered and added to L_{i+1}

Clicker



Suppose in BFS that the nodes in each layer are explored in a different order (e.g. reverse). Which of the following are true?

A. The nodes that appear in each layer may change

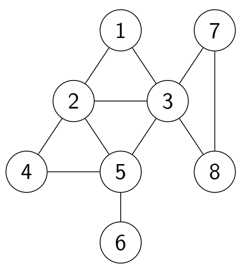
B. The BFS tree may change

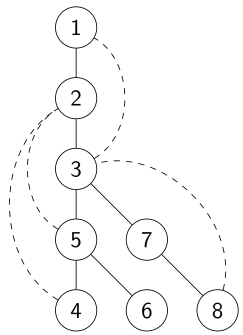
C. Both A and B

D. Neither A nor B

Depth-First Search

Depth-first search (DFS): keep exploring from the most recently added node until you have to backtrack.





Dotted edges: to already explored nodes

DFS: Recursive Implementation

DFS(u)
  mark u as "explored"
  for all edges (u, v) **do**
    if v is not "explored" **then**
      call DFS(v) recursively

DFS: Running Time

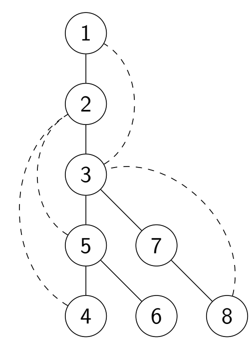
How to analyze if algorithm is recursive? Same: count executions of each line, *including* recursive call

DFS(u)	
mark u as "explored"	▷ n
for all edges (u, v) **do**	▷ $2m$
if v is not "explored" **then**	▷ $2m$
call DFS(v) recursively	▷ n

Running time: $O(m + n)$ same as BFS

DFS Tree

$T \leftarrow$ empty
DFS(u)
  mark u as "explored"
  for all edges (u, v) **do**
    if v is not "explored" **then**
      put (u, v) in T
      call DFS(v) recursively



Claim: Non-tree edges lead to (indirect) **ancestors**

DFS: Non-tree edges lead to ancestors

Claim: Let T be the tree discovered by DFS, and let (x, y) be an edge of G that is not in T. Then one of x or y is an ancestor of the other.

Proof:

- Let x be the first of the two nodes explored
- Is y explored at beginning of DFS(x)? No.
- At some point *during* DFS(x), we examine the edge (x, y). Is y explored then? Yes, otherwise we would put (x, y) in T
- $\Rightarrow$ y was explored *during* DFS(x)
- $\Rightarrow$ y is a descendant of x

Generic Traversal Implementations

Generic approach: maintain set of explored nodes and discovered nodes

- Explored = have seen this node and explored its outgoing edges
- Discovered = the "frontier". Have seen the node, but not explored its outgoing edges.

Generic Graph Traversal

Let A = data structure of discovered nodes

Traverse(s)

```
put s in A
while A is not empty do
    take a node v from A
    if v is not marked "explored" then
        mark v "explored"
        for each edge (v, w) incident to v do
            put w in A                          ▷ w is discovered
```

BFS: A is a queue (FIFO)   DFS: A is a stack (LIFO)

Clicker

```
put s in A
while A is not empty do
    take a node v from A
    if v is not marked "explored" then
        mark v "explored"
        for each edge (v, w) incident to v do
            put w in A                          ▷ w is discovered
```

Suppose we run this traveral code and every node is marked explored before it terminates. Which of the following is false?

A. Every node is marked "explored" exactly once.
B. A single node could be put into A more than once.
C. If $w \neq s$, the number of times that node w is put into A is degree(w).
D. It's possible that there exist nodes x and y with no path from x to y.

Exploring *all* Connected Components

How to explore entire graph even if it is disconnected?

while there is some unexplored node s **do**
  Traverse(s) ▷ Run BFS/DFS starting from s.
  Extract connected component containing s

Running time? Still $O(m + n)$

- Traversal of each component takes time proportional to the numbers of nodes $+$ edges in *that* component

Advice: usually OK to assume graph is connected. State if you are doing so and why it does not trivialize the problem.

Summary

- Graph traversal by BFS/DFS: basic algorithmic primitive used in many other algorithms
  - Is there a path from u to v?
  - Find all connected components
  - Produce trees with different properties, sometimes useful in algorithms
- $\Theta(m + n)$ time
- Different versions of general exploration strategy