COMPSCI 311: Introduction to Algorithms	**Fall 2022**

Discussion 8

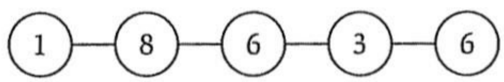
Your Name: ____________________ Collaborators: ____

You will be randomly assigned groups to work on these problems in discussion section.

Problem 1. Maximum Independent Set. *(Finish this problem from last week.)* Let $G = (V, E)$ be an undirected graph with n nodes. Recall that a subset of the nodes is called an **independent set** if no two of them are joined by an edge. Finding large independent sets is difficult in general; but here we'll see that it can be done efficiently if the graph is "simple" enough.

Call a graph $G = (V, E)$ a **path** if its nodes can be written as $v_1, v_2, \cdots, v_n$ with an edge between v_i and v_j if and only if the numbers i and j differ by exactly 1. With each node v_i, we associate a positive integer weight w_i.

Consider, for example, the following five-node path. The weights are the numbers drawn inside the nodes.



The goal in this question is to solve the following problem: *Find an independent set in a path G whose total weight is as large as possible.*

(a) What is the maximum-weight independent set in the example?

(b) Give an example to show that the following algorithm does not always find an independent set of maximum total weight.

Start with S equal to the empty set
while some node remains in G **do**
  Pick a node v_i of maximum weight
  Add v_i to S
  Delete v_i and its neighbors from G
return S

(c) Give an example to show that the following algorithm also does not always find an independent set of maximum total weight.

Let S_1 be the set of all v_i where i is an odd number
Let S_2 be the set of all v_i where i is an even number
(Note that S_1 and S_2 are both independent sets)
Determine which of S_1 or S_2 has greater total weight, and **return** this one

(d) Give an algorithm with $O(n)$ running time that takes an n-node path G with weights and returns the weight of the largest independent set.

(e) Now give an $O(n)$ algorithm to compute the maximum-weight independent set itself. Your algorithm should use values stored in the memoization array from your previous algorithm to trace back through the array and construct the optimal solution. (Hint: given the memoization arrary, how can you tell whether node v_n is in the optimal solution?)

Problem 2. Longest Increasing Subsequence. In the *longest increasing subsequence problem*, you are given as input an unsorted array A of length n, e.g,

$$A = 5, 2, 10, 3, -1, 6, 8, 9, 3$$

The goal is to find the longest strictly increasing subsequence of A. The subsequence need not be continguous. For example, the boxed numbers below indicate the longest increasing subsequence in our example:

$$A = 5, \boxed{2}, 10, \boxed{3}, -1, \boxed{6}, \boxed{8}, \boxed{9}, 3$$

To approach this problem, it is useful to define a "helper" function LIS(j) to compute the length of the longest increasing subsequence that *ends at* index j (i.e., it must *include* item $A[j]$ in the subsequence). Here are examples for $j = 3$ and $j = 5$:

$$5, \boxed{2}, \boxed{10} \qquad\qquad 5, 2, 10, 3, \boxed{\text{-1}}$$

Therefore LIS(3) should return 2, and LIS(5) should return 1.

Follow these steps to design a dynamic programming algorithm to find a longest increasing subsequence:

1. Write a recursive algorithm for LIS(j)
2. Translate this recursive algorithm into a recurrence. Define OPT(j) to be the length of the longest increasing subsequence ending at index j, and write a recurrence for OPT(j).
3. Use this recurrence to write an iterative algorithm to compute the value of OPT(j) and store it in the array entry $M[j]$ for all j.
4. Use the computed optimal values to find the value of the overall longest increasing subsequence (ending at any j).