COMPSCI 311: Introduction to Algorithms **Fall 2022**

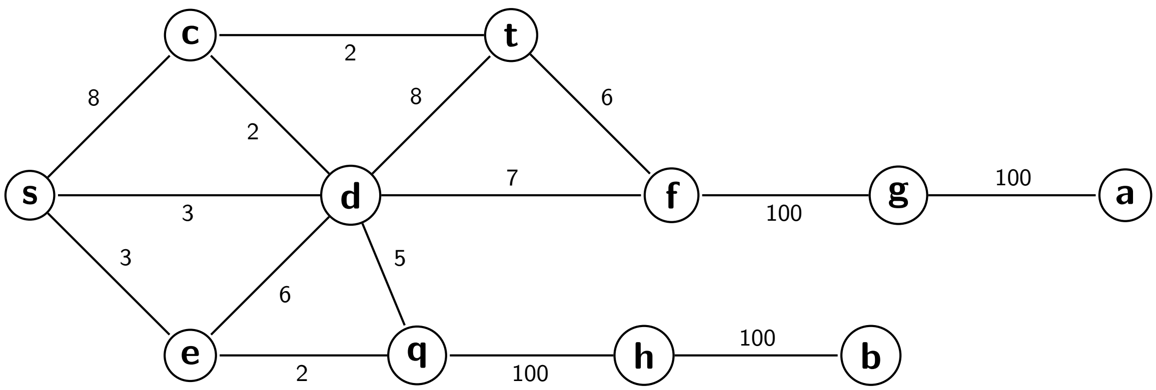
Discussion 5

Your Name: ____________________ Collaborators: ____

You will be randomly assigned groups to work on these problems in discussion section.

Problem 1. Dijkstra and negative edge weights.

(a) Execute Dijkstra's algorithm to find a shortest path from node s to rest of the nodes.



(b) Draw an edge between a and b with a weight of -1000. For which nodes, if any, does Dijkstra's algorithm find a different shortest path length? Does it find the correct shortest paths for every node? Is there such a thing as a shortest path between s and t in the new graph?

Problem 2. Construction. You're running a company contracted to build dorms at Mount Holyoke. Each dorm has a foundation, a living space, and a roof. These three components must be built in that order. The college has designed each dorm and each dorm component will need a certain amount of time to build; these times are known in advance.

Dorm	Foundation Time	Living Space Time	Roof Time
Anderson	2 days	2 days	3 days
Burleson	3 days	2 days	2 days
Ciesielski	4 days	3 days	2 days

Your company has only one excavator so it can only work on one foundation at any given time. However, the company has many different work crews, and can work on any number of living spaces and roofs in parallel with a foundation.

You want to build your dorms in an order such that the last dorm to finish being built will complete as soon as possible.

(a) Say your contract requires you to build Anderson, Burleson, and Ciesielski. What is an ordering where the last dorm to finish completes as soon as possible?

(b) Consider the general problem where thee are n dorms and for dorm i, the time to build the foundation, living space, and roof, respectively, are f_i, l_i, and r_i. Which of the following greedy rules produces an optimal solution?

- Order dorms by increasing foundation time f_i
- Order dorms by decreasing foundation time f_i
- Order dorms by increasing living space plus roof time $l_i + r_i$
- Order dorms by decreasing living space plus roof time $l_i + r_i$

(c) Now let's prove that the rule you selected is optimal by an exchange argument. Let A be the greedy solution. If O is an optimal solution and $O \neq A$, argue that you can modify O to get a new solution O' that is closer to A and no worse than O (so still optimal). Define an inversion as pair of dorms (i, j) that are out of order with respect to greedy solution. Define a *consecutive inversion* as a pair of dorms where i is scheduled immediately before j in O, but j comes before i in A.

- *True or false?* If O has an inversion, it has a consecutive inversion.

Now suppose that O is an optimal solution with a consecutive inversion i and j. If we swap i and j it is clear that we get a new schedule O' with one less inversion.

- Show that the overall finish time of O' is no later than the overall finish time of O.

This means that O' is still optimal. To complete the exchange argument, we note that there are at most $\binom{n}{2}$ inversions. Therefore, if O is not equal to A, we can apply this argument at most $\binom{n}{2}$ times to transform O into A *while preserving optimality at every step*, therefore proving that A is optimal.

(d) What is the running time of this algorithm?