Reasoning about programs

Last time

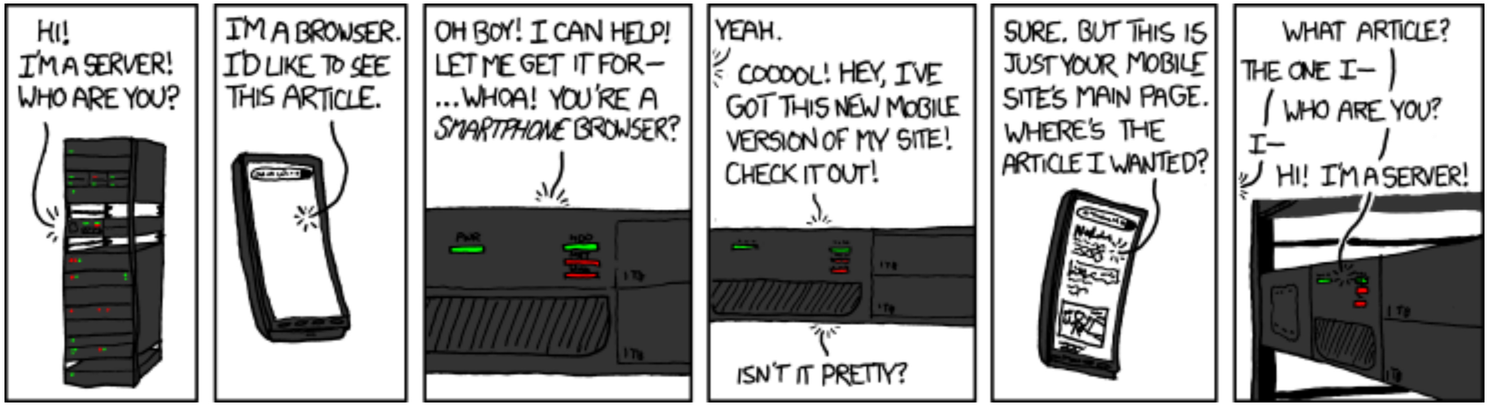
Coming up

- This Thursday, Nov 30:
  4th in-class exercise
  - sign up for group on moodle
  - bring laptop to class
- Final projects:
  - final project presentations: Tue Dec 12, in CS 150
    final submission due: Fri Dec 15, 11:55 PM

Project Final Presentations

- December 12, 10AM-11:15AM
- CS 150 (in the CS building)
- Think of this as a science fair.
- Each team will get an easel. Bring a poster or printed slides. And laptop for demo.
- Describe and discuss the solution, and demo the implementation.
- Will see (at least) 2 separate judges.
- Chance to see other projects too!

Reasoning about programs


HI! I'M A SERVER! WHO ARE YOU?
I'M A BROWSER. I'D LIKE TO SEE THIS ARTICLE.
OH BOY! I CAN HELP! LET ME GET IT FOR— ...WHOA! YOU'RE A *SMARTPHONE* BROWSER?
YEAH.
COOOOL! HEY, I'VE GOT THIS NEW MOBILE VERSION OF MY SITE! CHECK IT OUT!
ISN'T IT PRETTY?
SURE. BUT THIS IS JUST YOUR MOBILE SITE'S MAIN PAGE. WHERE'S THE ARTICLE I WANTED?
WHAT ARTICLE?
THE ONE I—
WHO ARE YOU?
I—
HI! I'M A SERVER!
PWR
HDD
1 TB

Ways to verify your code

- The hard way:
  - Make up some inputs
  - If it doesn't crash, ship it
  - When it fails in the field, attempt to debug
- The easier way:
  - Reason about possible behavior and desired outcomes
  - Construct simple tests that exercise that behavior
- Another way that can be easy
  - Prove that the system does what you want
    - Rep invariants are preserved
    - Implementation satisfies specification
  - Proof can be formal or informal (we will be informal)
  - Complementary to testing

Reasoning about code

- Determine what facts are true during execution
  - x > 0
  - for all nodes n: n.next.previous == n
  - array a is sorted
  - x + y == z
  - if x != null, then x.a > x.b
- Applications:
  - Ensure code is correct (via reasoning or testing)
  - Understand why code is incorrect

Forward reasoning

- You know what is true before running the code
  What is true after running the code?
- Given a precondition, what is the postcondition?

- Applications:
  Representation invariant holds before running code
  Does it still hold after running code?
- Example:

```
// precondition: x is even
x = x + 3;
y = 2x;
x = 5;
// postcondition:  ??
```

Backward reasoning

- You know what you want to be true after running the code

  What must be true beforehand in order to ensure that?
- Given a postcondition, what is the corresponding precondition?

- Applications:

  (Re-)establish rep invariant at method exit: what's required?

  Reproduce a bug: what must the input have been?
- Example:

  ```
  // precondition: ??
  x = x + 3;
  y = 2x;
  x = 5;
  // postcondition: y > x
  ```
- How did you (informally) compute this?

Forward vs. backward reasoning

- Forward reasoning is more intuitive for most people
  - Helps understand what will happen (simulates the code)
  - Introduces facts that may be irrelevant to goal

    Set of current facts may get large
  - Takes longer to realize that the task is hopeless
- Backward reasoning is usually more helpful
  - Helps you understand what should happen
  - Given a specific goal, indicates how to achieve it
  - Given an error, gives a test case that exposes it

Forward reasoning example

```
assert x >= 0;
i = x;
      // x ≥ 0  &  i = x
z = 0;
      // x ≥ 0  &  i = x  &  z = 0
while (i != 0) {        ⇐ What property holds here?
   z = z + 1;
   i = i –1;            ⇐ What property holds here?
}
      // x ≥ 0  &  i = 0  &  z = x
assert x == z;
```

Backward reasoning

Technique for backward reasoning:

- Compute the weakest precondition (wp)
- There is a wp rule for each statement in the programming language
- Weakest precondition yields strongest specification for the computation (analogous to function specifications)

Assignment

```
// precondition: ??
x = e;
// postcondition: Q
```

Precondition: Q with all (free) occurrences of x replaced by e

- Example:

```
// assert:  ??
x = x + 1;
// assert x > 0
```

Precondition = $(x+1) > 0$

Method calls

```
// precondition: ??
x = foo();
// postcondition: Q
```

- If the method has no side effects: just like ordinary assignment
- If it has side effects: an assignment to every variable it modifies

Use the method specification to determine the new value

If statements

```
// precondition:  ??
if (b) S1 else S2
// postcondition: Q
```

Essentially case analysis:

wp("if (b) S1 else S2", Q) =

$$\begin{aligned}(\quad & b \Rightarrow \text{wp}(\text{“S1”}, Q) \\ \wedge\ & \neg b \Rightarrow \text{wp}(\text{“S2”}, Q)\ )\end{aligned}$$

If: an example

```
// precondition: ??
if (x == 0) {
    x = x + 1;
} else {
    x = (x/x);
}
// postcondition:  x ≥ 0
```

Precondition:

wp("if (x==0) {x = x+1} else {x = x/x}", $x \geq 0$) =

= ($x = 0 \Rightarrow$ wp("x = x+1", $x \geq 0$)

& $x \neq 0 \Rightarrow$ wp("x = x/x", $x \geq 0$))

= $(x = 0 \Rightarrow x + 1 \geq 0)$ & $(x \neq 0 \Rightarrow x/x \geq 0)$

= $1 \geq 0$ & $1 \geq 0$

= true

Reasoning About Loops

- A loop represents an unknown number of paths
  - Case analysis is problematic
  - Recursion presents the same issue
- Cannot enumerate all paths
  - That is what makes testing and reasoning hard

Loops: values and termination

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

1) Pre-assertion guarantees that x ≥ y

2) Every time through loop

    x ≥ y holds and, if body is entered, x > y

    y is incremented by 1

    x is unchanged

    Therefore, y is closer to x (but x ≥ y still holds)

3) Since there are only a finite number of integers between x and y, y will eventually equal x

4) Execution exits the loop as soon as x = y

Understanding loops by induction

- We just made an inductive argument
  - Inducting over the number of iterations
- Computation induction
  - Show that conjecture holds if zero iterations
  - Assume it holds after n iterations and show it holds after n+1
- There are two things to prove:
  - Some property is preserved (known as “partial correctness”)
    - loop invariant is preserved by each iteration
  - The loop completes (known as “termination”)
    - The “decrementing function” is reduced by each iteration

Loop invariant for the example

```
// assert x ≥ 0 & y = 0
while (x != y) {
     y = y + 1;
}
// assert x = y
```

- So, what is a suitable invariant?
- What makes the loop work?

Loop Invariant (LI) = $x \geq y$

1) $x \geq 0$ & $y = 0 \Rightarrow$ LI
2) LI & $x \neq y$ {`y = y+1;`} LI
3) (LI & $\neg(x \neq y)$) $\Rightarrow$ $x = y$

Is anything missing?

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

Does the loop terminate?

Total Correctness via Well-Ordered Sets

- We have not established that the loop terminates
- Suppose that the loop always reduces some variable's value. Does the loop terminate if the variable is a
  - Natural number?
  - Integer?
  - Non-negative real number?
  - Boolean?
  - ArrayList?
- The loop terminates if the variable values are (a subset of) a well-ordered set
  - Ordered set
  - Every non-empty subset has least element

Decrementing Function

- Decrementing function D(X)
  - Maps state (program variables) to some well-ordered set
  - This greatly simplifies reasoning about termination

- Consider: `while (b) S;`
- We seek D(X), where X is the state, such that
  1. An execution of the loop reduces the function's value:
     $LI \ \& \ b \ \{\mathbf{S}\} \ D(X_{post}) < D(X_{pre})$
  2. If the function's value is minimal, the loop terminates:
     $(LI \ \& \ D(X) = minVal) \Rightarrow \neg b$

Proving Termination

```
// assert x ≥ 0 & y = 0
// Loop invariant: x ≥ y
// Loop decrements:  (x-y)
while (x != y) {
    y = y + 1;
}
// assert x = y
```

- Is “x-y” a good decrementing function?

1. Does the loop reduce the decrementing function’s value?
   // assert (y != x); let $d_{pre} = (x – y)$
   y = y + 1;
   // assert $(x_{post} – y_{post}) < d_{pre}$

2. If the function has minimum value, does the loop exit?
   $(x >= y \ \& \ x – y = 0) \rightarrow (x = y)$

Choosing Loop Invariant

- For straight-line code, the wp (weakest precondition) function gives us the appropriate property
- For loops, you have to guess:
  - The loop invariant
  - The decrementing function
- Then, use reasoning techniques to prove the goal property
- If the proof doesn't work:
  - Maybe you chose a bad invariant or decrementing function
    - Choose another and try again
  - Maybe the loop is incorrect
    - Fix the code
- Automatically choosing loop invariants is a research topic

In practice

I don’t routinely write loop invariants

I do write them when I am unsure about a loop and when I have evidence that a loop is not working

- Add invariant and decrementing function if missing
- Write code to check them
- Understand why the code doesn't work
- Reason to ensure that no similar bugs remain

More on Induction

- Induction is a very powerful tool

$$2^n = 1 + \sum_{k=1}^{n} 2^{k-1}$$

Proof by induction: Base Case

For n=1, $1 + \sum_{k=1}^{1} 2^{k-1} = 1 + 2^0 = 1 + 1 = 2 = 2^1$

Inductive Step

Assume $2^m = 1 + \sum_{k=1}^{m} 2^{k-1}$ and show that $2^{m+1} = 1 + \sum_{k=1}^{m+1} 2^{k-1}$

$$2^{m+1} = 1 + \sum_{k=1}^{m+1} 2^{k-1} = 1 + \sum_{k=1}^{m} 2^{k-1} + 2^m = 2^m + 2^m = 2 \times 2^m = 2^{m+1}$$

Is Induction Too Powerful?

Next time

- Using theorem provers to reason about programs
- We’ll use Z3
- Take a look at the tutorial before class: https://rise4fun.com/Z3/tutorial/guide