The Weird Warden

Solution

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Choose one prisoner to count signals. Each of the other 99 prisoners may send at most two signals. The counter announces success after counting 198.

A signal means turning an *off* light on. Agree on these rules before the visits begin:

1. Each non-counter keeps a private count of how many signals they have sent. If the light is off and they have sent fewer than two, they turn it on and increase their private count. Otherwise they leave it alone. They never turn it off.
2. Whenever the counter finds the light on, they turn it off and increase their count by one. If it is off, they do nothing. When their count reaches 198, they announce that everyone has visited.

Why 198 proves everyone has visited

Suppose some non-counter had never visited. The other 98 non-counters could have sent at most $98 \cdot 2 = 196$ signals. Even if the light was initially on, that adds at most one extra count. The counter could therefore have reached at most 197, not 198.

So a count of 198 proves that every non-counter has visited. The counter has certainly visited too.

Why the count will eventually reach 198

If the count is below 198, either a non-counter still has an unused signal or the light is on with a signal waiting to be counted. The fairness assumption says that every prisoner returns infinitely often. Whenever the light is on, the counter eventually visits and turns it off. Whenever it is off and signals remain, an eligible non-counter eventually visits and turns it on. Thus the remaining signals cannot be postponed forever.

The protocol needs no calendar or knowledge of the first day. It tolerates either initial light state. Using only one signal per non-counter would not give this protection: an initially on light could substitute for one prisoner's missing signal.

Bonus: how long?

There is no finite worst-case bound on the number of days. Even with independent uniform selection, the same prisoner might be selected for arbitrarily many days in a row. The original fairness condition alone does not specify an expected time.

Under the bonus assumption of independent uniform selection among the 100 prisoners each day, the expected times for *this protocol* are approximately:

Initial light	Expected days
Off	20,527.663
On	20,427.663

These are not claims of an optimal expected-time protocol. An unknown initial state is not automatically a 50–50 random initial state. The next page gives a reproducible calculation.

Calculating the expected time

Consider a moment when the light is off and the counter has not finished. Let a be the number of non-counters with two unused signals and b the number with one unused signal.

Let $\delta = 0$ if the initial light was off. Let $\delta = 1$ after an initially on light has been counted: that initial count means one of the 198 real signals may remain unused when the protocol stops.

Define $E_\delta(a, b)$ as the expected remaining number of days in this off-light state. The boundary condition is

$$E_\delta(a, b) = 0 \quad \text{when } 2a + b \leq \delta.$$

Otherwise, the expected wait for an eligible non-counter to turn the light on is $100/(a + b)$ days. Once it is on, the expected wait for the counter is 100 days. The next signal comes from a two-signal prisoner with probability $a/(a + b)$, or a one-signal prisoner with probability $b/(a + b)$. Hence

$$\begin{aligned} E_\delta(a, b) = {} & \frac{100}{a + b} + 100 \\ & + \frac{a}{a + b} E_\delta(a - 1, b + 1) + \frac{b}{a + b} E_\delta(a, b - 1). \end{aligned}$$

Omit a term whose numerator is zero; it need not refer to a state with a negative count.

The arguments on the right have fewer unused signals than the argument on the left, so the values can be calculated starting from the boundary.

An initially off light gives $E_0(99, 0)$. An initially on light first requires an average of 100 days for the counter to turn it off, giving $100 + E_1(99, 0)$. Evaluating the recurrence gives the numbers on the preceding page.

The expected time includes the day on which the final announcement is made.

History and sources: companion history PDF.