The Interesting Shape

Solution

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Gabriel's horn is one example.

Take the curve $y = 1/x$ for $x \geq 1$ and rotate it around the x-axis. This makes an infinitely long horn whose circular cross-sections keep shrinking. The solid can be written as

$$\{(x, y, z) : x \geq 1,\ y^2 + z^2 \leq 1/x^2\}.$$

At position x, its radius is $1/x$.

Why its volume is finite

A thin slice has cross-sectional area $\pi(1/x)^2$. Adding all the slices by integration gives

$$V = \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi.$$

The square of the shrinking radius decreases fast enough for this total to converge.

Why its surface area is infinite

A thin strip of the curved surface has circumference $2\pi/x$, multiplied by a slanted length that is at least the strip's horizontal length. Thus even the smaller comparison area diverges:

$$\text{curved area} \ \geq\ 2\pi \int_1^\infty \frac{1}{x}\, dx = \infty.$$

For completeness, the exact surface-area integral is

$$2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx,$$

which is at least the preceding divergent integral.

Adding a disk to cap the horn at $x = 1$ adds only a finite area and changes neither conclusion. The basic contrast is between adding terms that shrink like $1/x^2$, which can give a finite total, and terms that shrink only like $1/x$, which here do not.

History and sources: companion history PDF.