Solution to the Hard Problem

By Philip Thomas, January 2011 (Updated March 2014)

As best I can find, this puzzle is due to Michael O'Connor (`www.xorshammer.com`). A solution that is more easy to understand than my own can be found here: `http://cornellmath.wordpress.com/2007/09/13/the-axiom-of-choice-is-wrong/`. Also, my solution came from working with several others back in 2011, so I cannot claim it entirely as my own, beyond the wording.

Let $\mathcal{S}$ be the set of all possible hattings (each of which is a string of countably infinite binary numbers representing whether the ith person has a black or white hat). Let's make a flawed assumption, then come back and try to fix the mistake. Let's assume that there is a bijection from $\mathcal{S}$ to $\mathbb{N}$, the natural numbers. This is not true because $|\mathcal{S}| = \aleph_1$, while $|\mathbb{N}| = \aleph_0$. However, if such a bijection existed, we could number the hattings. Lets assume that the correct hatting number is h.

The ith individual sees the values of everyone else's hat, but does not know his or her own. Thus, he knows that the correct hatting number is either h_i^1 or h_i^2, where h_i^1 and h_i^2 denote identical hattings except that they differ on the color of the ith person's hat. Also notice that for every person, one of these two hattings is the correct one, h, and the other is unique. By unique, I mean that the ith person is the only one considering that hatting number, because it has the incorrect color for the ith person, which everyone else would have observed.

Let the ith person select the hat color that is consistent with the hatting with lesser value. That is, if $h_i^1 < h_i^2$ then the ith person selects the hat color he is assigned in the h_i^1th hatting. Otherwise he selects the color from the h_i^2th hatting. There are only $h - 1$ hattings with a smaller value than the correct hatting h, and because only one person could possibly guess each of these (remember that the incorrect guess that each person has is unique), at most $h-1$ people will guess incorrectly, as everyone else's lesser of h_i^1 and h_i^2 will be the correct hatting.

Ok, that's dandy, but we started with a false assumption—that there is a mapping from $\mathcal{S}$ to $\mathbb{N}$. To fix this, we assume the axiom of choice[1], which means that there exists a well-ordering for set $\mathcal{S}$. Thus, each person can select the least element from the set $\{h_i^1, h_i^2\}$. However, our solution remains broken because now there may be an infinite number of h_i^1 and h_i^2 that are less than h, so infinite people may guess incorrectly.

To fix this, we modify our strategy. First, we order the people (again with a well-ordering). We call each person's number his *rank*. Each person guesses that his hat is consistent with h_i, where h_i is now the least valued hatting (according to the well-ordering), that is consistent with the hat colors of all the people of higher rank. Let p_i denote the ith person. Notice that if $p_i < p_j$, then $h_i \geq h_j$, otherwise h_i would have been selected by p_j.

Also, if $h_i \neq h$ (by this we mean that h_i differs from h on the color of the ith person's hat) and $p_j < p_i$, then $h_j > h_i$. Notice that h_j is strictly greater than h_i. This is because h_i is the least hatting that is correct for all p_k, $p_k > p_i$. If it is wrong about the ith person's hat, then all p_j, $p_j < p_i$, will know that it is wrong, because they see that the jth person's hat does not match. Thus, when p_j selects the minimum hatting that is consistent with all p_k, $p_k > p_j$, it cannot select h_i, as it is incorrect, though it is the minimum choice for p_i. Thus, the new choice h_j must satisfy $h_j > h_i$.

[1] If you're not familiar with the axiom of choice of well-orderings, see the Wikipedia articles.

Next, consider the set of people who guess incorrectly, $\mathcal{S}'$. Order this set from lowest to highest rank. As you walk through this set of people, the hatting values they guessed must be strictly decreasing (see previous paragraph). According to the Wikipedia page on well-orderings, if a set is well ordered "Every decreasing sequence of elements of the set must terminate after only finitely many steps." Thus, the set of incorrect hatting guesses (at most one per person) made by the people who guessed incorrectly, must be finite.