The Duck and the Fox

Solution

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Let v be the duck's maximum swimming speed, with the fox's speed equal to 1 and the pond's radius equal to 1.

The duck can guarantee escape when $v > 0.2172336282\ldots$. This number is a threshold, not an attained slowest winning speed: at equality, the fox can prevent the duck from reaching the shore strictly before it. The exact answer uses calculus. First, here is a much simpler strategy that already beats $1/\pi$.

An easy strategy: get opposite, then go straight out

Swim on a small circle of radius $r < v$ centered on the pond. The duck's angular speed can be $v/r > 1$, faster than the fox's angular speed around the shore. Therefore the duck can get directly opposite the fox, even if the fox changes direction.

Then swim straight outward to shore. The duck has distance $1 - r$ to go, while the fox must run halfway around the pond, a distance π. The duck wins if

$$\frac{1-r}{v} < \pi.$$

A radius $r < v$ satisfying this inequality exists whenever

$$\boxed{v > \frac{1}{\pi + 1} \approx 0.241453.}$$

For example, a duck with speed $1/4$ can escape a fox four times as fast. Choose r just below $1/4$, get opposite, and dash out.

Doing better: keep turning while moving outward

The best route does not suddenly switch to a radial dash. It continues using part of the duck's speed to slow how quickly the fox closes the angular gap while moving toward shore.

In the ideal continuous-observation model, the duck can first reach radius v directly opposite the fox. Inside this radius it has enough angular speed to maintain opposition while moving outward. Write r for its distance from the center. Beyond $r = v$, split the duck's velocity into

$$\underbrace{\frac{v^2}{r}}_{\text{sideways speed, away from the fox}} \qquad \text{and} \qquad \underbrace{v\sqrt{1 - \frac{v^2}{r^2}}}_{\text{outward speed}}.$$

Their squared values add to v^2, so this respects the speed limit. "Away" means increasing the shorter angular separation from the fox. At exact opposition, the duck follows the fox's instantaneous direction to maintain the largest possible separation. This is a feedback rule, not a commitment to run in one fixed direction.

The next page explains the resulting threshold and checks that a still slower duck cannot guarantee escape.

The exact threshold (optional calculus)

Let θ be the smaller angle between the fox and duck as seen from the pond's center, so $0 \le \theta \le \pi$. Angles here are in radians. Define, for $r \ge v$,

$$g(r) = \sqrt{\frac{r^2}{v^2} - 1} - \arccos\left(\frac{v}{r}\right).$$

When the fox runs toward the duck's angular position, the outward-turning rule on the preceding page gives

$$\theta(r) = \pi - g(r).$$

The duck therefore reaches shore with a positive angular gap exactly when

$$\pi - \sqrt{\frac{1}{v^2} - 1} + \arccos v > 0.$$

The boundary between success and failure is the unique root in $(0, 1)$ of

$$\sqrt{\frac{1}{v^2} - 1} - \arccos v = \pi,$$

namely $v_* = 0.217233628211\ldots$. A fox can be almost $1/v_* = 4.603338849\ldots$ times as fast as the duck.

Why the formula holds. Let u_r be the duck's outward speed and u_t its sideways speed away from the fox. While $0 < \theta < \pi$, a chasing fox gives

$$\dot{\theta} = \frac{u_t}{r} - 1, \qquad \dot{r} = u_r.$$

For our chosen speeds, substitution and differentiation show that $\theta + g(r)$ stays constant. At $r = v$, it equals π, giving the displayed formula. If the fox runs the other way or slows down, the gap cannot be made smaller by that choice.

Why no other route permits a lower threshold. Differentiating g gives

$$g'(r) = \frac{\sqrt{r^2 - v^2}}{vr}.$$

Against a chasing fox, *any* allowed duck velocity satisfies

$$\frac{d}{dt}(\theta + g(r)) = \frac{u_t}{r} - 1 + g'(r)u_r \le 0,$$

because $u_r^2 + u_t^2 \le v^2$ and $(1/r)^2 + g'(r)^2 = 1/v^2$. This last step is the elementary fact that a dot product is at most the product of the two vector lengths.

Consider the duck's last departure from radius v before a proposed escape. Its angular separation is at most π. Hence it cannot reach shore with separation greater than $\pi - g(1)$. If the separation reaches zero earlier, the fox can track the duck's angle thereafter outside radius v, where the duck's angular speed is at most 1. Thus a nonpositive bound rules out guaranteed escape. At equality, simultaneous arrival is not escape under the question's rule.

The threshold and feedback formulation agree with the classical "Lady in the Lake" analysis discussed in Von Moll and Pachter (2024). The easy radial-dash strategy is sufficient but is not optimal.

History and sources: companion history PDF.