Solution to Othello Problem
By Philip Thomas, January 2011

We begin by numbering the colors from 0 to 6. Let h_i be the number for the ith person's hat. Let

$$n = \sum_{i=0}^{6} h_i. \tag{1}$$

Notice that nobody can observe n, as nobody knows his/her own hat number. Let

$$m = n \bmod 7. \tag{2}$$

We have the 0th person guess that $m = 0$, the 1st guess that $m = 1$ etc., so the ith person guesses $m = i$. Exactly one person is correct, while the others are all wrong.

So, if you are person i, and you want to guess that $m = i$, how many possible colors are there for your hat? Let

$$x_i = \left[\sum_{j=0}^{6} h_j \right] - h_i. \tag{3}$$

Notice that the ith person *can* compute x_i, by summing the hat values of everyone else. The ith person, who is trying to make $m = i$, must then select the value k such that $x_i + k \bmod (7) = i$. Because $0 \leq k < 7$, there is only one such k. Thus, there is only one possible hat selection for the ith person that is cosistent with everything s/he has seen, and which makes $m = i$. Each person guesses this value. Exactly one person will have assumed the correct m, and will guess his/her hat color correctly.

For example, if there were only two people with two hat colors, person 0 would try to make $m = 0$ (working modulo 2 now). So if the other person's hat color was 0, person 0 would guess color 0. If the other person's hat color was 1, person 0 would guess color 1, so $1 + 1 = 0 \bmod 2$. Person 1 would try to make $m = 1$. Thus, if person 0 has hat 0, person 1 guesses 1. If person 0 has hat 1, then person 1 guesses 0. Or, put more simply, person 0 guesses the same color as what he sees on person 1's head, and person 1 guesses the opposite color of what he sees on person 0's head. How do we know that one must be correct? Well, either both hats are the same color, or they are each different colors. In the first case, person 0 would guess his hat color correctly, while in the second case person 1 would guess correctly. The description above is an extension of this idea to more than 2 people.

Our solution is trivially extended to an arbitrary number of people, so long as the number of possible hat colors is equal to the number of people.