Mathematical Induction
Reading: EC 2.3

Peter J. Haas

INFO 150
Fall Semester 2018
Mathematical Induction

Overview
First Examples
Sentences About the Positive Integers
Formal Mathematical Induction
Examples
Strong versus Weak Induction
Overview

What is induction?

- Ordinary usage: infer the future based on the past ("the sun will rise tomorrow")
- Mathematical induction: prove the truth of the next integer, from the past
- Exploit fact that recurrence relations are often relatively simple
- The perfect tool for attacking complexity

Example

- $1 + 2 + \cdots + 100 = 5,050$
- $1 + 2 + \cdots + 100 + 101 =$?
- Since first answer is correct:

\[
1 + 2 + \cdots + 100 + 101 = (1 + 2 + \cdots + 100) + 101 \\
= (5,050) + 101 = 5,151
\]
Induction as a Game

Example

- \( a_k = a_{k-1} + (2k - 1) \) with \( a_1 = 1 \)
- Try some values: \( a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_5 = 25 \)
- It looks like \( a_n = n^2 \), but can you prove this?
- Try a Big Honking Table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>Formula for ( a_n )</th>
<th>Value of ( a_n )</th>
<th>( n^2 )</th>
<th>Is ( a_n = n^2 )?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a_1 = 1 )</td>
<td>1</td>
<td>1</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>( a_2 = a_1 + (2 \cdot 2 - 1) )</td>
<td>1 + 3 = 4</td>
<td>4</td>
<td>yes</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>49</td>
<td>( a_{49} = a_{48} + (2 \cdot 49 - 1) )</td>
<td>2304 + 97 = 2401</td>
<td>2401</td>
<td>yes</td>
</tr>
<tr>
<td>50</td>
<td>( a_{50} = a_{49} + (2 \cdot 50 - 1) )</td>
<td>2401 + 99 = 2500</td>
<td>2500</td>
<td>yes</td>
</tr>
</tbody>
</table>

Look at the last row: \( a_{50} = a_{49} + (2 \cdot 50 - 1) = 49^2 + (2 \cdot 50 - 1) \)

Generalize: \( a_m = (m - 1)^2 + (2 \cdot m - 1) \)
Proposition

The sequence defined recursively by $a_1 = 1$ and $a_k = a_{k-1} + (2k - 1)$ has closed form $a_n = n^2$.

Proof:

1. Check every row in the table up through $m - 1$
2. For the $m$th row, $a_m = a_{m-1} + (2m - 1)$
3. Since we have checked row $m - 1$, we know that $a_{m-1} = (m - 1)^2$
4. Substitute and solve:

   $$a_m = a_{m-1} + (2m - 1) = (m - 1)^2 + (2m - 1) = m^2 - 2m + 1 + 2m - 1 = m^2,$$

   and so row $m$ of the table checks out.
5. We therefore know that, repeating this procedure, every row will check out.
### Example, Continued

A tabular view of the general case

<table>
<thead>
<tr>
<th>$n$</th>
<th>Formula for $a_n$</th>
<th>Value of $a_n$</th>
<th>$n^2$</th>
<th>Is $a_n = n^2$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1 = 1$</td>
<td>1</td>
<td>1</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>$a_2 = a_1 + (2 \cdot 2 - 1)$</td>
<td>$1 + 3 = 4$</td>
<td>4</td>
<td>yes</td>
</tr>
<tr>
<td>3</td>
<td>$a_3 = a_2 + (2 \cdot 3 - 1)$</td>
<td>$4 + 5 = 9$</td>
<td>9</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m - 1$</td>
<td>$a_{m-1} = a_{m-2} + (2 \cdot (m - 1) - 1)$</td>
<td>$(m - 1)^2$</td>
<td>$(m - 1)^2$</td>
<td>yes</td>
</tr>
<tr>
<td>$m$</td>
<td>$a_m = a_{m-1} + (2 \cdot m - 1)$</td>
<td>$m^2$</td>
<td>$m^2$</td>
<td>??</td>
</tr>
</tbody>
</table>

Example: given row $n = 29$, check row $n = 30$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Formula for $a_n$</th>
<th>Value of $a_n$</th>
<th>$n^2$</th>
<th>Is $a_n = n^2$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>29</td>
<td>$a_{29} = a_{28} + (2 \cdot 29 - 1)$</td>
<td>$29^2$</td>
<td>$29^2$</td>
<td>yes</td>
</tr>
</tbody>
</table>

$$a_{30} = a_{29} + (2 \cdot 30 - 1) = 29^2 + (2 \cdot 30 - 1) = 841 + 59 = 900 = 30^2$$  ✓
Another Example

Recursive sequence: \( a_n = a_{n-1} + 2 \cdot n \) with \( a_1 = 2 \)

Proposed closed form: \( a_n = n(n + 1) \)

Tabular setup

<table>
<thead>
<tr>
<th>( n )</th>
<th>Formula for ( a_n )</th>
<th>Value of ( a_n )</th>
<th>( n(n + 1) )</th>
<th>Is ( a_n = n(n + 1) )?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a_1 = 2 )</td>
<td>2</td>
<td>1 \cdot 2</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>( a_2 = a_1 + 2 \cdot 2 )</td>
<td>2 + 4 = 6</td>
<td>2 \cdot 3</td>
<td>yes</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( m - 1 )</td>
<td>( a_{m-1} = a_{m-2} + 2 \cdot (m - 1) )</td>
<td>( (m - 1)m )</td>
<td>( (m - 1)m )</td>
<td>yes</td>
</tr>
<tr>
<td>( m )</td>
<td>( a_m = a_{m-1} + 2 \cdot m )</td>
<td>?</td>
<td>( m(m + 1) )</td>
<td>???</td>
</tr>
</tbody>
</table>

Check row \( m \):

\[
a_m = a_{m-1} + 2 \cdot m = (m-1)m + 2 \cdot m = (m-1+2)m = m(m+1)
\]
Closed Formulas for Sums

Claim: \( \sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \sum_{i=1}^{n} i ) or ( 1 + 2 + \cdots + n )</th>
<th>Simplified sum</th>
<th>( \frac{n(n+1)}{2} )</th>
<th>( \text{sum} = \frac{n(n+1)}{2} ? )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sum_{i=1}^{1} i = 1 )</td>
<td>1</td>
<td>( \frac{1.2}{2} = \frac{2}{2} = 1 )</td>
<td>Yes</td>
</tr>
<tr>
<td>2</td>
<td>( \sum_{i=1}^{2} i = 1 + 2 )</td>
<td>1 + 2 = 3</td>
<td>( \frac{2.3}{2} = \frac{6}{2} = 3 )</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>( \sum_{i=1}^{3} i = (1 + 2) + 3 )</td>
<td>3 + 4 = 6</td>
<td>( \frac{3.4}{2} = \frac{12}{2} = 6 )</td>
<td>Yes</td>
</tr>
<tr>
<td>4</td>
<td>( \sum_{i=1}^{4} i = (1 + 2 + 3) + 4 )</td>
<td>6 + 4 = 10</td>
<td>( \frac{4.5}{2} = \frac{20}{2} = 10 )</td>
<td>Yes</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>34</td>
<td>( \sum_{i=1}^{34} i = (1 + 2 + \cdots + 33) + 34 )</td>
<td>561 + 34 = 595</td>
<td>( \frac{34.35}{2} = \frac{595}{2} = 297.5 )</td>
<td>Yes</td>
</tr>
<tr>
<td>35</td>
<td>( \sum_{i=1}^{35} i = (1 + 2 + \cdots + 34) + 35 )</td>
<td>595 + 35 = 630</td>
<td>( \frac{35.36}{2} = \frac{630}{2} = 315 )</td>
<td>Yes</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( m-1 )</td>
<td>( \sum_{i=1}^{m-1} i = 1 + 2 = \cdots + (m - 1) )</td>
<td>( \frac{(m-1)m}{2} )</td>
<td>( \frac{(m-1)m}{2} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( m )</td>
<td>( \sum_{i=1}^{m} i = 1 + 2 = \cdots + m )</td>
<td>???</td>
<td>???</td>
<td>???</td>
</tr>
</tbody>
</table>

Check row \( m \):
\[
\sum_{i=1}^{m} i = \frac{(m)(m+1)}{2}
\]

Check row 146:
\[
\sum_{i=1}^{146} i = \frac{(146)(147)}{2} = \frac{21318}{2} = 10659
\]

Lecture 9
Sentences About the Positive Integers

Definition

A statement about the positive integers is a predicate \( P(n) \) with the set of positive integers as its domain.

Example: Which are statements about the positive integers?

- \( n^2 + n \) is even \( \checkmark \)
- \( 100 - n \) \( \times \)
- \( 100 - n > 83 \) \( \checkmark \)
- John has fewer than \( n \) apples in his refrigerator \( \checkmark \)

Example: For each predicate, write the sentence when \( n = 2 \) and \( n = 30 \) and determine whether it is true or false

- \( E(n) \) is the statement “\( n^2 + n \) is even”
  - \( E(2) \): “\( 6 \) is even” \( \checkmark \)
  - \( E(30) \): “\( 930 \) is even” \( \checkmark \)
- \( G(n) \) is the statement “\( 100 - n > 83 \)”
  - \( G(2) \): “\( 98 > 83 \)” \( \checkmark \)
  - \( G(30) \): “\( 70 > 83 \)” \( \times \)
- \( S(n) \) is the statement “\( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \)”
  - \( S(30) \): “\( 1 + 12 + \cdots + 30 = \frac{30(31)}{2} \)” \( \checkmark \)
Sentences, Continued

Example

- \( P(n) \) is “If there are \( n \) students in the class, the room will be too small”
- What is \( P(35) \)? If there are 35 students in the class, the room will be too small
- What is \( P(m - 1) \)? If there are \( m - 1 \) students in the class, the room will be too small

Example

- \( S(n) \) is “1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}”
- Rewrite using \( \Sigma \) notation
  \[
  S(n) : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
  \]
- Write \( S(1) \), \( S(2) \), and \( S(3) \) and determine if true or false
  \[
  S(1) : \sum_{i=1}^{1} i^2 = \frac{1^2}{6} \neq 1.2 \times 3 \times 1 \Rightarrow \text{False}
  \]
  \[
  S(2) : \sum_{i=1}^{2} i^2 = 1^2 + 2^2 = \frac{2 \times 3 \times 5}{6} \Rightarrow \text{True}
  \]
  \[
  S(3) : \sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = \frac{6 \times 3 \times 7}{6} \Rightarrow \text{True}
  \]
- Write \( S(m - 1) \) and simplify it
  \[
  S(m - 1) : \sum_{i=1}^{m-1} i^2 = \frac{(m-1)(m)(2m-1)}{6}
  \]