Mathematical Writing
Reading: EC 2.1

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INFO 150
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Mathematical Writing

Overview
Translating Between English and Math
Review of Implications and Their Contrapositives
Mathematical Proofs
Tracing a Proof
Simple Proofs About Numbers
Overview

Goal: Learn to write mathematically

- We’ll first study properties of common mathematical objects (starting with integers)
- We’ll learn how to present mathematical proofs about these properties to others
- We’ll focus on inductive proofs, the most common type
A Couple of Definitions

Definition 1
A positive integer \( n > 1 \) is prime if it cannot be factored as \( n = a \cdot b \), where both \( a \) and \( b \) are greater than 1.

Definition 2
A perfect square is a positive integer that is equal to \( z^2 \) for some positive integer \( z \).
Translating Between English and Math

Unlike English, most math statement are implications

- In English: “Whenever an object has property $P$ then it must have property $Q$”
- In mathspeak: “if $p$, then $q$” $p \rightarrow q$
- English allows a wide variety of equivalent forms

Example: rewrite into “if, then form”

- Whenever $n$ is an even integer, $2n^3 + n$ is divisible by 3
  
  $\text{If an integer } n \text{ is even, then } 2n^3 + n \text{ is div. by 3}$

- For every prime $n$, $n^2 - n + 41$ is prime
  
  $\text{If } n \text{ is prime, then } n^2 - n + 41 \text{ is prime}$ (counterexample: $n=41$)

- The sum of the interior angles in any triangle is 180°
  
  $\text{If } t \text{ is a triangle, then sum of } t\text{'s interior angles } = 180°$

Observations

- Not every mathematical statement is true
  (Which is the false statement?)
- Not every mathematical statement is about numbers
Review of Implications and Their Contrapositives

The law: “if you are drinking beer, then you are at least 21 years of age”

The cards (see picture)

Which cards must Jones turn over to check that everyone is obeying the law?

Must be because someone is drinking beer and under 21

I.e., “if \( p \), then \( q \)” is false only if \( p \) is true and \( q \) is false

So trooper is looking for a counterexample

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<table>
<thead>
<tr>
<th>Hypothesis ((p))</th>
<th>Conclusion ((q))</th>
<th>Implication ((\text{If } p, \text{ then } q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>You are drinking beer</td>
<td>You are at least 21</td>
<td>You are obeying the law</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
Example (more abstract):

- Four cards, each with a letter on one side and a number on the other
- Claim: if one side has a vowel, then the other side has an odd number
- Which cards do you need to check?

Recall contrapositives

- Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- A proposition (or a predicate) and its contrapositive are logically equivalent
- Example: “If you are under 21 years of age, then you are not drinking beer”
Implications and Contrapositives, Continued

Ex: Describe a counterexample; are the pairs contrapositives?

1(a) If \( n \) is even, then \( n! + 1 \) is prime
1(b) If \( n \) is odd, then \( n! + 1 \) is not prime

2(a) If \( n^2 \) is even, then \( n \) is even
2(b) If \( n \) is not even, then \( n^2 \) is not even

3(a) If \( n \) is prime, then the number following \( n \) is not a perfect square
3(b) If \( m \) is a perfect square, then the number preceding \( m \) is not prime

If the counterexamples are the same, then the propositions are contrapositives
Mathematical Proofs

Trooper Jones proves that the law is being obeyed
- Jones makes sure there are no counterexamples (p true and q false)
- Easy, since at most 4 people to check (and some of them don't need checking)
- This procedure holds true in general

Example: play the role of Trooper Jones
1. For every integer \( n \geq 1 \), if \( n \) is odd, then \( n^2 + 4 \) is a prime number
2. For every positive integer \( n \), if \( n \) is odd, then \( n^3 - n \) is divisible by 4

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n^2 + 4 )</th>
<th>Prime?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>yes</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>no</td>
</tr>
<tr>
<td>5</td>
<td>29</td>
<td>yes</td>
</tr>
<tr>
<td>7</td>
<td>53</td>
<td>yes</td>
</tr>
<tr>
<td>9</td>
<td>85</td>
<td>no</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n^3 - n )</th>
<th>Divisible by 4?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>( 0 = 4 \times 0 )</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td>( 24 = 6 \times 24 )</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>( 120 = 4 \times 30 )</td>
</tr>
<tr>
<td>7</td>
<td>336</td>
<td>( 336 = 4 \times 84 )</td>
</tr>
<tr>
<td>9</td>
<td>720</td>
<td>( 720 = 4 \times 180 )</td>
</tr>
</tbody>
</table>

Observations
- No need to check even numbers (hypothesis is false)
- If you haven't found a counterexample yet, that doesn't mean there isn't one
The essence of a proof

- You will never find a counterexample
- Equivalently, no matter what number is chosen that satisfies the hypothesis, it is guaranteed to also satisfy the conclusion

Proof as a game between Author and Reader

1. Reader chooses a value of $n$ that satisfies the hypothesis
2. Author tries to demonstrate that conclusion is true for this value of $n$
3. If conclusion is true for this choice of $n$, Author is successful & Reader takes another turn
4. If conclusion is false for this choice of $n$, Reader wins

Observation

- If statement is true, then the game never ends
- So Author writes an argument to convince Reader that game will never end
- This argument is a mathematical proof
- Author and Reader must agree on the meaning of all terms in the statement
First Example

Informal statement
Other than 3, 4 there is no pair of consecutive integers where the first is a prime number and the second is a perfect square.

Theorem
For all integers $n > 4$, if $n$ is a perfect square, then $n - 1$ is not a prime number.

Some sample plays of the game:

<table>
<thead>
<tr>
<th>Reader’s $n$</th>
<th>Author’s factorization</th>
<th>Prime?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4^2 = 16$</td>
<td>$15 = 3 \times 5$</td>
<td>no</td>
</tr>
<tr>
<td>$6^2 = 36$</td>
<td>$35 = 5 \times 7$</td>
<td>no</td>
</tr>
<tr>
<td>$7^2 = 49$</td>
<td>$48 = 6 \times 8$</td>
<td>no</td>
</tr>
<tr>
<td>$10^2 = 100$</td>
<td>$99 = 9 \times 11$</td>
<td>no</td>
</tr>
<tr>
<td>$12^2 = 144$</td>
<td>$143 = 11 \times 13$</td>
<td>no</td>
</tr>
</tbody>
</table>
First Example, Continued

Pattern of the game
Reader chooses $n = m^2$, then Author tries to factor $n - 1$

Recall: $m^2 - 1 = (m - 1)(m + 1)$

Informal proof
Every time you choose a perfect square (greater than 4) for $n$, say, $n = m^2$ ($m$ a positive integer), I can factor $n - 1$. This is because $n - 1$ is the same as $m^2 - 1$, which factors as $(m - 1)(m + 1)$. As long as these factors are both at least 2—which they are since $n > 4$—this will demonstrate that $n - 1$ is not prime.

Formal proof
Let a perfect square $n > 4$ be given. By definition of a perfect square, $n = m^2$ for some positive integer $m$. Since $n > 4$, it follows that $m > 2$. Now the number $n - 1 = m^2 - 1$ can be factored as $(m - 1)(m + 1)$. Since $m > 2$, then both $m - 1$ and $m + 1$ are greater than 1, so $(m - 1)(m + 1)$ is a factorization of $n - 1$ into the product of two positive numbers, each greater than 1. By the definition of a prime number, it follows that $n - 1$ is not prime.
Tracing a Proof

<table>
<thead>
<tr>
<th>$n = m^2$</th>
<th>$n - 1$</th>
<th>$(m - 1)(m + 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = (3)^2$</td>
<td>8</td>
<td>$(3 - 1)(3 + 1) = (2)(4)$</td>
</tr>
<tr>
<td>$n = (4)^2$</td>
<td>15</td>
<td>$(4 - 1)(4 + 1) = (3)(5)$</td>
</tr>
<tr>
<td>$n = (7)^2$</td>
<td>48</td>
<td>$(7 - 1)(7 + 1) = (6)(8)$</td>
</tr>
<tr>
<td>$n = (10)^2$</td>
<td>99</td>
<td>$(10 - 1)(10 + 1) = (9)(11)$</td>
</tr>
<tr>
<td>$n = (12)^2$</td>
<td>143</td>
<td>$(12 - 1)(12 + 1) = (11)(13)$</td>
</tr>
<tr>
<td>$n = (25)^2$</td>
<td>624</td>
<td>$(25 - 1)(25 + 1) = (24)(26)$</td>
</tr>
</tbody>
</table>

Note:

- A trace helps you understand a proof, it is not a proof itself
- A trace can help you detect flaws in faulty proofs
Some More (Precise) Definitions

Definition 1

An integer is **even** if it can be written in the form \( n = 2 \cdot K \) for some integer \( K \). An integer \( m \) is **odd** if it can be written in the form \( n = 2 \cdot L + 1 \) for some integer \( L \).

Definition 2

An integer is **divisible by 4** if it can be written in the form \( n = 4 \cdot M \) for some integer \( M \).

Closure property of the integers

Whenever the operations of addition, subtraction, or multiplication are applied to integers, the result is an integer.

Example: Use the definitions to show the following

- 72, 0, and -18 are even
- 81 and -15 are odd
- 72 is divisible by 4
- For any choice of integer \( n \), \( 4n^2 - 2n \) is even

\[
\begin{align*}
72 &= 2 \times 36, \quad 0 = 2 \cdot 0, \quad -18 = 2 \cdot (-9) \\
81 &= 2 \times 40 + 1, \quad -15 = 2 \cdot (\text{odd}) + 1 \\
72 &= 4 \times 18 \\
4n^2 - 2n &= 2 \cdot (2n^2 - n)
\end{align*}
\]

\( \mathbb{Z}_{>0} \) not closed under subtraction

\( \mathbb{Z} \) not closed under division
Another Example

Proposition
The result of summing any odd integer with any even integer is an odd integer.

Proof
1. Let odd integer $x$ and even integer $y$ be given.
2. By the definition of “odd”, there exists an integer $A$ such that $x = 2 \cdot A + 1$.
3. By the definition of “even”, there exists an integer $B$ such that $y = 2 \cdot B$.
4. This means that

$$x + y = (2 \cdot A + 1) + 2 \cdot B$$
$$= 2 \cdot A + 2 \cdot B + 1$$
$$= 2 \cdot (A + B) + 1.$$  

5. Since $A + B$ is an integer (by the closure property), $x + y$ can be written as 2 times an integer plus 1, so by the definition of “odd”, $x + y$ is odd.
Tracing the Proof

**Proof for** $x = 17$ **and** $y = 12$

1. By the definition of “odd”, there exists an integer $A$ ($A = 8$) such that $x = 2 \cdot A + 1$ ($17 = 2 \cdot 8 + 1$).
2. By the definition of “even”, there exists an integer $B$ ($B = 6$) such that $y = 2 \cdot B$ ($12 = 2 \cdot 6$).
3. This means that

$$17 + 12 = (2 \cdot 8 + 1) + 2 \cdot 6 = 2 \cdot 8 + 2 \cdot 6 + 1 = 2 \cdot (8 + 6) + 1.$$ 

4. So $17 + 12 = 29$ can be written as 2 times an integer (14) plus 1, so by the definition of “odd”, $17 + 12$ is odd.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$A$</th>
<th>$B$</th>
<th>$x+y$</th>
<th>$2 \cdot (A+B) + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>12</td>
<td>8</td>
<td>6</td>
<td>29</td>
<td>2 \cdot (14) + 1</td>
</tr>
<tr>
<td>37</td>
<td>8</td>
<td>18</td>
<td>4</td>
<td>45</td>
<td>2 \cdot (22) + 1</td>
</tr>
<tr>
<td>101</td>
<td>14</td>
<td>50</td>
<td>7</td>
<td>115</td>
<td>2 \cdot (57) + 1</td>
</tr>
<tr>
<td>-17</td>
<td>84</td>
<td>-9</td>
<td>42</td>
<td>67</td>
<td>2 \cdot (33) + 1</td>
</tr>
<tr>
<td>51</td>
<td>50</td>
<td>25</td>
<td>25</td>
<td>101</td>
<td>2 \cdot (50) + 1</td>
</tr>
</tbody>
</table>
An Error to Avoid

An incorrect proof

1. By the definition of “odd”, there exists an integer $A$ such that $x = 2 \cdot A + 1$.
2. By the definition of “even”, there exists an integer $A$ such that $y = 2 \cdot A$.
3. This means that

$$x + y = (2 \cdot A + 1) + 2 \cdot A = 2 \cdot A + 2 \cdot A + 1 = 2 \cdot (A + A) + 1.$$ 

4. So $x + y$ can be written as 2 times an integer plus 1, so by the definition of “odd”, $x + y$ is odd.

Try tracing the proof with $x = 17$ and $y = 12$

- Then $A = 8$ since $17 = 2 \cdot 8 + 1$?
- But then proof says that $17 + 12 = 2 \cdot 16 + 1$ (false!)
- Maybe $A = 6$, since $12 = 2 \cdot 6$?
- But then proof says that $17 + 12 = 2 \cdot 12 + 1$ (false!)
- Yuck.

Moral: In any proof, use different variables to represent different things

- We use $A$ and $B$ because the two numbers are not known to be the same
Proposition

The sum of two even integers is even.

1. given \( x \) even and \( y \) even
2. \( x = 2A \) and \( y = 2B \) by def. of "even"
3. \( x + y = 2A + 2B = 2(A + B) \)
4. \( A + B \) is an integer (closure)
   so \( x + y \) is even by def. of "even"
Proposition

If \( n \) is even, then \( n^2 \) is divisible by 4.

Proof (see textbook for a “letter to the reader” format)

1. Let an even integer \( n \) be given.
2. By the definition of “even”, there exists an integer \( k \) at \( n = 2 \cdot k \).
3. This means that

\[
    n^2 = (2 \cdot k)^2 = 4k^2 = 4 \cdot (k^2).
\]

4. Since \( k \) is an integer, \( k^2 \) is an integer, so \( n^2 \) can be written as 4 times an integer, so by the definition of “divisible by 4”, \( n^2 \) is divisible by 4.
Another Pitfall

Proposition

If $n^2$ is even, then $n$ is even.

Flawed proof

1. We can write $n^2 = 2k$ for some integer $k$.
2. Divide both sides by $n$, getting $n = 2 \cdot (k/n)$.
3. Since $k/n$ is an integer, this proves the result.

What is the problem here? Try square roots?

A trick: Sometimes proving the contrapositive is easier

- Formal statement: For all integers $n$, if $n^2$ is even, then $n$ is even
- Contrapositive: For all integers $n$, if $n$ is odd, then $n^2$ is odd
The Final Theorem and Proof

Proposition
For all integers \( n \), if \( n \) is odd, then \( n^2 \) is odd

Proof
1. Let odd integer \( n \) be given.
2. By definition of “odd”, \( n = 2k + 1 \) for some integer \( k \)
3. Then

\[
n^2 = (2k + 1)^2
\]
\[
= 4k^2 + 4k + 1
\]
\[
= 2 \cdot (2k^2 + 2k) + 1.
\]

4. Since \( 2k^2 + 2k \) is an integer, this proves that \( n^2 \) can be written as 2 times an integer plus 1, so \( n^2 \) is odd.