Mathematical Writing
Overview
Translating Between English and Math
Review of Implications and Their Contrapositives
Mathematical Proofs
Tracing a Proof
Simple Proofs About Numbers
Overview

**Goal: Learn to write mathematically**

- We’ll first study properties of common mathematical objects (starting with integers)
- We’ll learn how to present mathematical proofs about these properties to others
- We’ll focus on **inductive** proofs, the most common type
Definition 1

A positive integer \( n > 1 \) is prime if it cannot be factored as \( n = a \cdot b \), where both \( a \) and \( b \) are greater than 1.

Definition 2

A perfect square is a positive integer that is equal to \( z^2 \) for some positive integer \( z \).
Translating Between English and Math

Unlike English, most math statement are implications
  ▶ In English: “Whenever an object has property \( P \) then it must have property \( Q \)”
  ▶ In mathspeak: “if \( p \), then \( q \)” \( p \rightarrow q \)
  ▶ English allows a wide variety of equivalent forms

Example: rewrite into “if, then form”
  ▶ Whenever \( n \) is an even integer, \( 2n^3 + n \) is divisible by 3
    \[ \text{If an integer is even, then } 2n^3 + n \text{ is div. by } 3 \]
  ▶ For every prime \( n \), \( n^2 - n + 41 \) is prime
    \[ \text{If } n \text{ is prime, then } n^2 - n + 41 \text{ is prime} \]
  ▶ The sum of the interior angles in any triangle is 180°
    \[ \text{If } t \text{ is a triangle, then the sum of } t's \text{ interior angles is } 180° \]

Observations
  ▶ Not every mathematical statement is true (2nd statement is false) \( n = 41 \)
  ▶ Not every mathematical statement is about numbers
Review of Implications and Their Contrapositives

Trooper Jones in the Pub

- The law: “if you are drinking beer, then you are at least 21 years of age”
- Law is broken if someone is drinking beer and under 21
- I.e., “if $p$, then $q$” is false only if $p$ is true and $q$ is false
- So trooper is looking for a counterexample

<table>
<thead>
<tr>
<th>Hypothesis ($p$)</th>
<th>Conclusion ($q$)</th>
<th>Implication (If $p$, then $q$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>You are drinking beer</td>
<td>You are at least 21</td>
<td>You are obeying the law</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Lecture 6
Recall contrapositives

- Contrapositive of \( p \rightarrow q \) is \( \neg q \rightarrow \neg p \)

- A proposition (or a predicate) and its contrapositive are logically equivalent

- Example:

  Implication: “If you are drinking beer, then you are at least 21 years of age, ”

  Contrapositive: “If you are under 21 years of age, then you are not drinking beer”
Mathematical Proofs

Trooper Jones proves that the law is being obeyed

- Jones makes sure there are no counterexamples ($p$ true and $q$ false)
- Easy, since at most 4 people to check (and some of them don't need checking)
- This procedure holds true in general

Example: play the role of Trooper Jones

1. For every integer $n \geq 1$, if $n$ is odd, then $n^2 + 4$ is a prime number
2. For every positive integer $n$, if $n$ is odd, then $n^3 - n$ is divisible by 4

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n^2 + 4$</th>
<th>Prime?</th>
<th>$n$</th>
<th>$n^3 - n$</th>
<th>Divisible by 4?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>Y</td>
<td>1</td>
<td>0</td>
<td>$0 \cdot 4 = 0$</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>Y</td>
<td>3</td>
<td>24</td>
<td>$6 \cdot 4 = 24$</td>
</tr>
<tr>
<td>5</td>
<td>29</td>
<td>Y</td>
<td>5</td>
<td>120</td>
<td>$30 \cdot 4 = 120$</td>
</tr>
<tr>
<td>7</td>
<td>53</td>
<td>Y</td>
<td>7</td>
<td>336</td>
<td>$44 \cdot 4 = 336$</td>
</tr>
<tr>
<td>9</td>
<td>85</td>
<td>N</td>
<td>9</td>
<td>720</td>
<td>$180 \cdot 4 = 720$</td>
</tr>
</tbody>
</table>

Observations

- No need to check even numbers
- If you haven't found a counterexample yet, that doesn't mean there isn't one
The essence of a proof

- You will never find a counterexample
- Equivalently, no matter what number is chosen that satisfies the hypothesis, it is guaranteed to also satisfy the conclusion
Mathematical Proofs as Games

The essence of a proof

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- Equivalently, no matter what number is chosen that satisfies the hypothesis, it is guaranteed to also satisfy the conclusion

Proof as a game between Author and (Skeptical) Reader

1. Reader chooses a value of $n$ that satisfies the hypothesis
2. Author tries to demonstrate that conclusion is true for this value of $n$
3. If conclusion is true for this choice of $n$, Author is successful & Reader takes another turn
4. If conclusion is false for this choice of $n$, Reader wins
The essence of a proof

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2. Author tries to demonstrate that conclusion is true for this value of $n$
3. If conclusion is true for this choice of $n$, Author is successful & Reader takes another turn
4. If conclusion is false for this choice of $n$, Reader wins

Observation

- If statement is true, then the game never ends
- So Author writes an argument to convince Reader that game will never end
- This argument is a mathematical proof
- Author and Reader must agree on the meaning of all terms in the statement
First Example

Informal statement

Other than 3, 4 there is no pair of consecutive integers where the first is a prime number and the second is a perfect square.
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Theorem

For all integers $n > 4$, if $n$ is a perfect square, then $n - 1$ is not a prime number.
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For all integers $n > 4$, if $n$ is a perfect square, then $n - 1$ is not a prime number.

Some sample plays of the game:

<table>
<thead>
<tr>
<th>Reader’s $n$</th>
<th>Author’s factorization</th>
<th>Prime?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4^2 = 16$</td>
<td>$15 = 3 \times 5$</td>
<td>no</td>
</tr>
<tr>
<td>$6^2 = 36$</td>
<td>$35 = 5 \times 7$</td>
<td>no</td>
</tr>
<tr>
<td>$7^2 = 49$</td>
<td>$48 = 6 \times 8$</td>
<td>no</td>
</tr>
<tr>
<td>$10^2 = 100$</td>
<td>$99 = 9 \times 11$</td>
<td>no</td>
</tr>
<tr>
<td>$12^2 = 144$</td>
<td>$143 = 11 \times 13$</td>
<td>no</td>
</tr>
</tbody>
</table>
Pattern of the game

Reader chooses $n = m^2$, then Author tries to factor $n - 1$
Pattern of the game
Reader chooses \( n = m^2 \), then Author tries to factor \( n - 1 \)

Recall: \( m^2 - 1 = (m - 1)(m + 1) \)
First Example, Continued

Pattern of the game

Reader chooses $n = m^2$, then Author tries to factor $n - 1$

Recall: $m^2 - 1 = (m - 1)(m + 1)$

Informal proof

Every time you choose a perfect square (greater than 4) for $n$, say, $n = m^2$ ($m$ a positive integer), I can factor $n - 1$. This is because $n - 1$ is the same as $m^2 - 1$, which factors as $(m - 1)(m + 1)$. As long as these factors are both at least 2—which they are since $n > 4$—this will demonstrate that $n - 1$ is not prime.
First Example, Continued

Pattern of the game

Reader chooses $n = m^2$, then Author tries to factor $n - 1$

Recall: $m^2 - 1 = (m - 1)(m + 1)$

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Formal proof

Let a perfect square $n > 4$ be given. By definition of a perfect square, $n = m^2$ for some positive integer $m$. Since $n > 4$, it follows that $m > 2$. Now the number $n - 1 = m^2 - 1$ can be factored as $(m - 1)(m + 1)$. Since $m > 2$, then both $m - 1$ and $m + 1$ are greater than 1, so $(m - 1)(m + 1)$ is a factorization of $n - 1$ into the product of two positive numbers, each greater than 1. By the definition of a prime number, it follows that $n - 1$ is not prime.
Tracing a Proof

<table>
<thead>
<tr>
<th>$n = m^2$</th>
<th>$n - 1$</th>
<th>$(m - 1)(m + 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = (3)^2$</td>
<td>8</td>
<td>$(3 - 1)(3 + 1) = (2)(4)$</td>
</tr>
<tr>
<td>$n = (4)^2$</td>
<td>15</td>
<td>$(4 - 1)(4 + 1) = (3)(5)$</td>
</tr>
<tr>
<td>$n = (7)^2$</td>
<td>48</td>
<td>$(7 - 1)(7 + 1) = (6)(8)$</td>
</tr>
<tr>
<td>$n = (10)^2$</td>
<td>99</td>
<td>$(10 - 1)(10 + 1) = (9)(11)$</td>
</tr>
<tr>
<td>$n = (12)^2$</td>
<td>143</td>
<td>$(12 - 1)(12 + 1) = (11)(13)$</td>
</tr>
<tr>
<td>$n = (25)^2$</td>
<td>624</td>
<td>$(25 - 1)(25 + 1) = (24)(26)$</td>
</tr>
</tbody>
</table>

**Note:**

- A trace helps you understand a proof, it is **not** a proof itself
- A trace can help you detect flaws in faulty proofs
Some More (Precise) Definitions

Definition 1
An integer is **even** if it can be written in the form \( n = 2 \cdot K \) for some integer \( K \). An integer \( m \) is **odd** if it can be written in the form \( n = 2 \cdot L + 1 \) for some integer \( L \).

Definition 2
An integer is **divisible by 4** if it can be written in the form \( n = 4 \cdot M \) for some integer \( M \).

Closure property of the integers
Whenever the operations of addition, subtraction, or multiplication are applied to integers, the result is an integer.

Example: Use the definitions to show the following
- 72, 0, and -18 are even
- 81 and -15 are odd
- 72 is divisible by 4
- For any choice of integer \( n \), \( 4n^2 - 2n \) is even

\[
72 = 2 \cdot 36 \quad 0 = 2 \cdot 0 \quad -18 = 2 \cdot -9
\]
\[
81 = 2 \cdot 40 + 1 \quad (L = 40 \text{ in the definition})
\]
\[
72 = 4 \cdot 18 \quad (M = 18 \text{ in the definition})
\]
\[
4n^2 - 2n = 2 \cdot (2n^2 - n) \quad 2n^2 - n \text{ is a integer, by closure}
\]
Another Example

Proposition

The result of summing any odd integer with any even integer is an odd integer.

Proof

1. Let odd integer $x$ and even integer $y$ be given.
2. By the definition of “odd”, there exists an integer $A$ such that $x = 2 \cdot A + 1$.
3. By the definition of “even”, there exists an integer $B$ such that $y = 2 \cdot B$.
4. This means that

$$x + y = (2 \cdot A + 1) + 2 \cdot B = 2 \cdot A + 2 \cdot B + 1 = 2 \cdot (A + B) + 1.$$

5. Since $A + B$ is an integer (by the closure property), $x + y$ can be written as 2 times an integer plus 1, so by the definition of “odd”, $x + y$ is odd.
Tracing the Proof

Proof for \( x = 17 \) and \( y = 12 \)

1. By the definition of “odd”, there exists an integer \( A (A = 8) \) such that \( x = 2 \cdot A + 1 \) \((17 = 2 \cdot 8 + 1)\).

2. By the definition of “even”, there exists an integer \( B (B = 6) \) such that \( y = 2 \cdot B \) \((12 = 2 \cdot 6)\).

3. This means that

\[
17 + 12 = (2 \cdot 8 + 1) + 2 \cdot 6 = 2 \cdot 8 + 2 \cdot 6 + 1 = 2 \cdot (8 + 6) + 1.
\]

4. So \( 17 + 12 = 29 \) can be written as 2 times an integer (14) plus 1, so by the definition of “odd”, \( 17 + 12 \) is odd.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( A )</th>
<th>( B )</th>
<th>( x + y )</th>
<th>( 2 \cdot (A + B) + 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>12</td>
<td>8</td>
<td>6</td>
<td>29</td>
<td>( 2 \cdot (14) + 1 )</td>
</tr>
<tr>
<td>37</td>
<td>8</td>
<td>18</td>
<td>4</td>
<td>45</td>
<td>( 2 \cdot (22) + 1 )</td>
</tr>
<tr>
<td>101</td>
<td>14</td>
<td>50</td>
<td>7</td>
<td>115</td>
<td>( 2 \cdot (57) + 1 )</td>
</tr>
<tr>
<td>-17</td>
<td>84</td>
<td>-9</td>
<td>42</td>
<td>67</td>
<td>( 2 \cdot (33) + 1 )</td>
</tr>
<tr>
<td>51</td>
<td>50</td>
<td>25</td>
<td>25</td>
<td>101</td>
<td>( 2 \cdot (50) + 1 )</td>
</tr>
</tbody>
</table>
An Error to Avoid

An incorrect proof

1. By the definition of “odd”, there exists an integer $A$ such that $x = 2 \cdot A + 1$.
2. By the definition of “even”, there exists an integer $A$ such that $y = 2 \cdot A$.
3. This means that

$$x + y = (2 \cdot A + 1) + 2 \cdot A = 2 \cdot A + 2 \cdot A + 1 = 2 \cdot (A + A) + 1.$$ 

4. So $x + y$ can be written as 2 times an integer plus 1, so by the definition of “odd”, $x + y$ is odd.

Try tracing the proof with $x = 17$ and $y = 12$

- Then $A = 8$ since $17 = 2 \cdot 8 + 1$?
  - But then proof says that $17 + 12 = 2 \cdot 16 + 1$ (false!)
  - Maybe $A = 6$, since $12 = 2 \cdot 6$?
  - But then proof says that $17 + 12 = 2 \cdot 12 + 1$ (false!)
  - Yuck.

Moral: In any proof, use different variables to represent different things

- We use $A$ and $B$ because the two numbers are not known to be the same
Proposition

The sum of two even integers is even.

1. Let \( x \) and \( y \) be two given, even integers.
2. By definition of even, \( x = 2 \cdot \alpha \) and \( y = 2 \cdot \beta \) for two integers \( \alpha \) and \( \beta \).
3. \( x + y = 2 \cdot \alpha + 2 \cdot \beta = 2 \cdot (\alpha + \beta) \).
4. By closure, \( \alpha + \beta \) is an integer.
5. Therefore \( x + y = 2 \cdot (\text{integer}) \).
6. By definition of "even", \( x + y \) is even.
Another Example

Proposition

If \( n \) is even, then \( n^2 \) is divisible by 4.

Proof (see textbook for a “letter to the reader” format)

1. Let an even integer \( n \) be given.
2. By the definition of “even”, there exists an integer \( k \) such that \( n = 2 \cdot k \).
3. This means that
   \[
   n^2 = (2 \cdot k)^2 = 4k^2 = 4 \cdot (k^2).
   \]
4. Since \( k \) is an integer, \( k^2 \) is an integer, so \( n^2 \) can be written as 4 times an integer, so by the definition of “divisible by 4”, \( n^2 \) is divisible by 4.
Another Pitfall

Proposition

If \( n^2 \) is even, then \( n \) is even.

Flawed proof

1. We can write \( n^2 = 2k \) for some integer \( k \).
2. Divide both sides by \( n \), getting \( n = 2 \cdot (k/n) \).
3. Since \( k/n \) is an integer, this proves the result.

What is the problem here? Try square roots?

A trick: Sometimes proving the contrapositive is easier

- Formal statement: For all integers \( n \), if \( n^2 \) is even, then \( n \) is even
- Contrapositive: For all integers \( n \), if \( n \) is odd, then \( n^2 \) is odd
The Final Theorem and Proof

Proposition

For all integers $n$, if $n$ is odd, then $n^2$ is odd

Proof

1. Let odd integer $n$ be given.
2. By definition of “odd”, $n = 2k + 1$ for some integer $k$
3. Then

\[ n^2 = (2k + 1)^2 \]
\[ = 4k^2 + 4k + 1 \]
\[ = 2 \cdot (2k^2 + 2k) + 1. \]

4. Since $2k^2 + 2k$ is an integer, this proves that $n^2$ can be written as 2 times an integer plus 1, so $n^2$ is odd.