Predicates
Reading: EC 1.4

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Predicates

Simple Predicates and Their Negations
Predicates and Sets
Quantified Predicates
Negating Quantified Predicates
Multiple Quantifiers and Their Negation
Simple Predicates

Definition

A predicate \( P(x) \) is a statement having a variable \( x \) such that whenever \( x \) is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write \( P(x_1, x_2, \ldots) \).

Example 1: \( P(n) = "n \text{ is even}" \)

- \( n = 2: P(2) = "2 \text{ is even}" \ [P(2) = T] \)
- \( n = 17: P(17) = "17 \text{ is even}" \ [P(17) = F] \)

Example 2: Evaluate each of the following predicates for \( x = 2, 23, -5, 15 \)

- \( P(x) = "x > 15": P(2) = "2 > 15" = F, P(23) = T, P(-5) = F, P(15) = F \)
- \( Q(x) = "x \leq 15": Q(2) = T, Q(23) = F, Q(-5) = T, Q(15) = T \)
- \( R(x) = "(x > 5) \land (x < 20)": R(2) = F, R(23) = F, R(-5) = F, R(15) = T \)
Negation of Simple Predicates

Techniques carry over from negation of propositions

Example:

<table>
<thead>
<tr>
<th>$P(x)$</th>
<th>$\neg P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &gt; 5$</td>
<td>$\neg(x &gt; 5) \equiv x \leq 5$</td>
</tr>
<tr>
<td>$(x &gt; 0) \land (x &lt; 10)$</td>
<td>$(x \leq 0) \lor (x \geq 10)$</td>
</tr>
<tr>
<td>$\neg(x = 8)$</td>
<td>$(x = 8)$</td>
</tr>
<tr>
<td>$(x \geq 0) \lor (y \geq 0)$</td>
<td>$\neg(x &gt; 0) \land \neg(y &lt; 0) = (x \leq 0) \land (y \geq 0)$ actually this is of the form $P(x,y)$</td>
</tr>
</tbody>
</table>

Evaluate $P(1, 2) = T$ $P(-1, 3) = T$ $P(-7, -2) = F$
Informal Definition

A set is a collection of objects, which are called elements or members.

Example: \( D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)

- For each predicate, list elements that make it true, and similarly for the negation

<table>
<thead>
<tr>
<th>( P(x) )</th>
<th>True for . . .</th>
<th>( \neg P(x) )</th>
<th>True for . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \geq 8 )</td>
<td>( 8, 9, 10 )</td>
<td>( x &lt; 8 )</td>
<td>( 1, 2, 3, 4, 5, 6, 7 )</td>
</tr>
<tr>
<td>( (x &gt; 5) \land (x \text{ is even}) )</td>
<td>( 6, 8, 10 )</td>
<td>( (x \leq 5) \lor (x \text{ is odd}) )</td>
<td>( 1, 2, 3, 4, 5, 7, 9 )</td>
</tr>
<tr>
<td>( x^2 = x ) (evenly) ( (x + 1) ) is divisible by 3</td>
<td>( 1 )</td>
<td>( x \neq x ) ( (x+1) ) is not div. by 3</td>
<td>( 1, 3, 4, 5, 6, 7, 8, 9 )</td>
</tr>
<tr>
<td>( x &gt; 0 )</td>
<td>( 2, 5, 8 )</td>
<td>( x \leq 0 )</td>
<td>( 3, 4, 5, 7, 9, 10 )</td>
</tr>
<tr>
<td>( x &gt; x^2 )</td>
<td>none of ( D )</td>
<td>none of ( D )</td>
<td>none of ( D )</td>
</tr>
</tbody>
</table>

We call \( D \) the domain of the predicate
Truth and Quantifiers

Example: $D = \{-1, 0, 1, 2\}$

<table>
<thead>
<tr>
<th>$P(x)$</th>
<th>True for these members of $D$</th>
<th>True for at least one?</th>
<th>True for all?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; 0$</td>
<td>$-1$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$x \geq -3$</td>
<td>$-1, 0, 1, 2$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$x^2 &lt; x$</td>
<td>none of $D$</td>
<td>$\times$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$x^2 \geq x$</td>
<td>all of $D$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

Examples of statements with quantifiers

- For every $k$ that is a member of the set $A = \{1, 2, 3, 4, 5\}$, it is true that $k < 20$
- There exists a member $m$ of the set $G = \{-1, 0, 1\}$ such that $m^2 = m$

Quantifier notation

- $\in$: “in” or “belonging to” (set membership)
- $\forall$: “for all” or “for every”
- $\exists$: “there is (at least one)” or “there exists (at least one)”

Rewrite the prior statements using mathematical notation

- $\forall k \in A, k < 20$
- $\exists m \in G, m^2 = m$

much shorter 😊
Quantified Predicates

Definitions

**Quantified predicate:** A predicate with one or more quantifiers

**Counterexample:** Example showing that a “for all” statement is false

**Example 1:** Translate from English to math and assess truth, for $D = \{3, 4, 5, 10, 20, 25\}$
- For every $n$ that is a member of $D$, $n < 20$:
- For all $n$ in the set $D$, $n < 5$ or $n$ is a multiple of 5:
- There is (at least one) $k$ in the set $D$ such that $k^2$ is also in $D$:
- There exists $m$ a member of the set $D$ such that $m \geq 3$:

**Example 2:** Translate from math to English and assess truth, for $D = \{-2, -1, 0, 1, 2\}$
- $\forall n \in D$, $n > -2$: For every $n$ in the set $D$, $n > -2$ (False)
- $\exists n \in D$, $n > -2$: For one or more $n$ in the set $D$, $n > -2$ (True) $\frac{-1}{2}$
- $\forall n \in D$, $(n > -3) \land (n < 3)$: For every $n$ in the set $D$, $n > -3$ and $n < 3$ (True) $\frac{-1}{2}$
- $\forall x \in D$, $x^2 \leq 4$: For every $x$ in the set $D$, $x^2 \leq 4$ (True)
- $\exists m \in D$, $m > 10$: For at least one $m$ in the set $D$, $m > 10$ (False)
Specify the Domain!

For each quantified statement, determine the domain $D$ and rewrite formally. If the
domain is ambiguous, give examples of how different domains can change the truth
of the statement.

- $\mathbb{R} =$ the real numbers ($\mathbb{R}_{>0} =$ positive real numbers)
- $\mathbb{Z} =$ the integers, i.e., $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$

1. For all $x$, $x^2 \geq x$
   - If $D = \mathbb{R}$: $\forall x \in \mathbb{R}, x^2 \geq x$ [false since $x = 0.5$ is a counterexample]
   - If $D = \mathbb{Z}$: $\forall x \in \mathbb{Z}, x^2 \geq x$ [true]

2. $\forall$ even integer $m$, $m$ ends in the digit 0, 2, 4, 6, or 8
   - $D =$ set of even integers: $\forall m \in D$, $m$ ends in the digit 0, 2, 4, 6, or 8

3. There is an integer $n$ whose square root is also an integer
   - $D = \mathbb{Z}$: $\exists k \in \mathbb{Z}$, $\sqrt{k} \in \mathbb{Z}$

4. Every integer greater than 0 has a square that is greater than 0
   - $D = \mathbb{R}_{>0}$: $\forall n \in \mathbb{R}_{>0}$, $n^2 > 0$
Example: For $D = \{-2, -1, 0, 1, 2\}$, explain why each predicate is false. Write the negation in English and formally.

1. $\forall d \in D, d < -2$: $p(2) = F$, so there exists $d \in D$ with $d \geq -2$ (i.e., a counterexample)
   
   **negation:** $\exists d \in D, d \geq -2$

2. $\exists m \in D, m > 10$: $m \leq 10$ for all $m \in D$, so $p(m) = F$ for all $m \in D$

   **negation:** $\forall m \in D, m \leq 10$
Negating Quantified Statements in General

**Proposition**

1. The negation of $\forall x \in D, \, P(x)$ is $\exists x \in D, \, \neg P(x)$
2. The negation of $\exists x \in D, \, Q(x)$ is $\forall x \in D, \, \neg Q(x)$

**Example:** For $D = \{-1, 0, 1, 2\}$, write the negation & determine which version is true

1. $\forall x \in D, \, (x \leq 0) \lor (x \geq 2)$: $\exists x \in D, \, (x > 0) \land (x < 2)$ is True (De Morgan's law)
2. $\exists x \in D, \, (x < 0) \lor (x^2 > 0)$: $\forall x \in D, \, (x > 0) \land (x^2 \leq 0)$ is False
3. $\forall x \in D, \, x^2 < x$: $\exists x \in D, \, x^2 \geq x$ is True
4. There exists $x \in D$ such that $x^2 < x$: $\forall x \in D, \, x^2 \geq x$ is True
Multiple Quantifiers

Reminder: Predicates can have multiple arguments

Example: \( P(x, y) = (x \in \mathbb{Z}) \land (y \in \mathbb{Z}) \land (x \cdot y = 36) \)

- Evaluate: \( P(9, 4) = \top \) \( P(-6, -6) = \top \) \( P(4, -1) = \bot \)
- If we replace \( \mathbb{Z} \) by \( \mathbb{R} \), then \( P(x, y) = \top \) for infinitely many \((x, y)\) pairs (e.g., \( x = 72, y = 0.5 \))

Multiple quantifiers of the same type (the easier case)

- There exist integers \( x \) and \( y \) such that \( x \cdot y = 36 \)
  - \( \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y = 36 \) or \( \exists x, y \in \mathbb{Z}, x \cdot y = 36 \) \( \text{True, e.g., } x = y = 6 \)
- For all integers \( x \) and \( y \), it is true that \( x \cdot y = 36 \)
  - \( \forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y = 36 \) or \( \forall x, y \in \mathbb{Z}, x \cdot y = 36 \) \( \text{False} \) \( \text{Counterexample: } x = 3, y = 2 \)
Write the following as quantified statements and assess whether each is true

- There are odd integers \( m \) and \( n \) whose product is 35:
  \[ \exists m, n \text{ odd integers such that } mn = 35 \] (True: \( m = 7, n = 5 \))

- There are even integers \( m \) and \( n \) whose product is 35:
  \[ \exists m, n \text{ even integers such that } mn = 35 \] (False)

- For every choice of integers \( s \) and \( t \), it is true that \( s^2 + t^2 \geq 0 \):
  \[ \forall s, t \text{ integers } s^2 + t^2 \geq 0 \] (True)

- For every choice of real numbers \( x \), \( y \), and \( z \), we have \( x + y + z > 1 \):
  \[ \forall x, y, z \in \mathbb{R}, x + y + z = 1 \] (False, Counterexample: \( x = -5, y = 0, z = 1 \))
Mixed Quantifiers

For two variables: Two basic kinds

- \( \forall x, \exists y, P(x, y) \)  
  Opponent gives you \( x \), you need to find \( y \) to make predicate true
- \( \exists y, \forall x, P(x, y) \)  
  You choose \( y \) so that predicate is true for any \( x \) that opponent gives you

Versus ambiguous English sentences

- “For every problem there is a solution” vs “There is a solution for every problem”  
  Two ways of saying the same thing
- Let \( P(x, y) = “x \text{ is a solution for problem } y” \)
- \( \forall y, \exists x, P(x, y) \) vs \( \exists x, \forall y, P(x, y) \)  
  There is one solution to all problems

Playing the Truth Game: Which of the following are true?

1. \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3 \)  
   Counterexample: \( x = 0 \)
   \( y \text{ choose } y < 15 - x \)

2. \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 15 \)  
   \( y \text{ choose } y = 15 - x \)

3. \( \exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 15 \)  
   For any \( y \) that you pick, e.g., \( y = 3 \), opponent can find \( x \) st. \( x + y < 15 \), e.g., \( x = 2 \)
Negating Multiple Quantifiers

Apply our proposition from left to right:

- The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$
- The negation of $\exists x \in D, Q(x)$ is $\forall x \in D, \neg Q(x)$

Example 1

$$\neg(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3)$$ initial negation

$$\exists x \in \mathbb{Z}, \neg(\exists y \in \mathbb{Z}, x + 2y = 3)$$ by proposition

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(x + 2y = 3)$$ by proposition

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + 2y \neq 3)$$ equivalent form of “not equal”

Example 2

$$\neg(\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x + y = 13) \land (x \cdot y = 36))$$ initial negation

$$\forall x \in \mathbb{Z}, \neg(\exists y \in \mathbb{Z}, (x + y = 13) \land (x \cdot y = 36))$$ by proposition

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg((x + y = 13) \land (x \cdot y = 36))$$ by proposition

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(x + y = 13) \lor \neg(x \cdot y = 36)$$ DeMorgan’s laws

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + y \neq 13) \lor (x \cdot y \neq 36)$$ equivalent form of “not equal”
Negating Multiple Quantifiers: Examples

Negate each quantified predicate. Which is true, the predicate or its negation?

\[ \forall x \in \mathbb{R}_{>0}, \exists y \in \mathbb{R}, (y > x) \land (x + y = 2x) : \]
\[ \exists x \in \mathbb{R}_{>0}, \forall y \in \mathbb{R}, (y > x) \land (x + y > 2x) : \]
\[ \exists x \in \mathbb{R}_{>0}, \forall y \in \mathbb{R}, (y \leq x) \lor (x + y \neq 2x) : \]
\[ \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y \leq 0 : \]
\[ \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y = 0 \]
\[ \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x - y) \leq 0 \]
\[ \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y \neq 0 \]
\[ \forall x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 \geq 0 \]
\[ \exists x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 < 0 \]