Predicates
Reading: EC 1.4

Peter J. Haas

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Predicates

Simple Predicates and Their Negations
Predicates and Sets
Quantified Predicates
Negating Quantified Predicates
Multiple Quantifiers and Their Negation
Simple Predicates

Definition

A **predicate** $P(x)$ is a statement having a variable $x$ such that whenever $x$ is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write $P(x_1, x_2, \ldots)$. 

Example 1: $P(n) = \text{\textquotedblleft}n \text{ is even\textquotedblright}$

- $P(2) = \text{\textquotedblleft}2 \text{ is even\textquotedblright}$
- $P(17) = \text{\textquotedblleft}17 \text{ is even\textquotedblright}$

Example 2: Evaluate the following predicate for $x = 2, 23, 5, 15$

$IR(x) = \text{\textquotedblleft}(x > 5)^\land (x < 20)\text{\textquotedblright}$
Simple Predicates

Definition

A predicate $P(x)$ is a statement having a variable $x$ such that whenever $x$ is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write $P(x_1, x_2, \ldots)$.

Example 1: $P(n) = \text{"n is even"}$

- $n = 2$: $P(2) = \text{"2 is even"}$ [$P(2) = T$]
- $n = 17$: $P(17) = \text{"17 is even"}$ [$P(17) = F$]

Example 2: Evaluate the following predicate for $x = 2, 23, 5, 15$:

$I_R(x) = \text{"(x > 5) ^ (x < 20)"}$
Simple Predicates

Definition

A predicate $P(x)$ is a statement having a variable $x$ such that whenever $x$ is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write $P(x_1, x_2, \ldots)$.

Example 1: $P(n) = \text{“}n\text{ is even}\text{”}$

- $n = 2$: $P(2) = \text{“}2\text{ is even}\text{”} \ [P(2) = T]$
- $n = 17$: $P(17) = \text{“}17\text{ is even}\text{”} \ [P(17) = F]$

Example 2: Evaluate the following predicate for $x = 2, 23, -5, 15$

- $R(x) = \text{“(}x > 5\text{) \land (}x < 20\text{)”}: \ R(2) = F \quad R(23) = F$
- $R(-5) = F \quad R(15) = T$
Negation of Simple Predicates

Techniques carry over from negation of propositions

Example:

<table>
<thead>
<tr>
<th>$P(x)$</th>
<th>$\neg P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &gt; 5$</td>
<td>$\neg(x &gt; 5) \equiv x \leq 5$</td>
</tr>
<tr>
<td>$(x &gt; 0) \land (x &lt; 10)$</td>
<td>$(x \leq 0) \lor (x \geq 10)$</td>
</tr>
<tr>
<td>$\neg(x = 8)$</td>
<td>$x = 8$</td>
</tr>
</tbody>
</table>

Equivalent for all values of $x$

DeMorgan’s Law

Double negation

Example 2: $P(x, y) = (x \geq 0) \lor (y \geq 0)$

- Negate $P(x, y)$: $(x < 0) \land (y < 0)$
- Evaluate $P(1, 2) = T$ $P(-1, 3) = T$ $P(-7, -2) = F$
Predicates and Sets

Informal Definition

A set is a collection of objects, which are called elements or members.

Example: \( D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)

- For each predicate, list elements that make it true, and similarly for the negation

<table>
<thead>
<tr>
<th>( P(x) )</th>
<th>True for ( \ldots )</th>
<th>( \neg P(x) )</th>
<th>True for ( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \geq 8 )</td>
<td>8, 9, 10</td>
<td>( x &lt; 8 )</td>
<td>1, 2, 3, 4, 5, 6, 7</td>
</tr>
<tr>
<td>( (x &gt; 5) \land (x \text{ is even}) )</td>
<td>6, 8, 10</td>
<td>( (x \leq 5) \lor (x \text{ is odd}) )</td>
<td>1, 2, 3, 4, 5, 7, 9</td>
</tr>
<tr>
<td>( x^2 = x )</td>
<td>1</td>
<td>( \neg x \neq x )</td>
<td>( 2, 5, 8 )</td>
</tr>
<tr>
<td>( (x + 1) \text{ is divisible by } 3 )</td>
<td>( 2, 5, 8 )</td>
<td>( (x + 1) \text{ not div. by } 3 )</td>
<td>( 1, 2, 3, 4, 5, 7, 9, 10 )</td>
</tr>
<tr>
<td>( x &gt; 0 )</td>
<td>all</td>
<td>( x \leq 0 )</td>
<td>( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 )</td>
</tr>
<tr>
<td>( x &gt; x^2 )</td>
<td>none</td>
<td>( x \leq x^2 )</td>
<td>none</td>
</tr>
</tbody>
</table>

We call \( D \) the domain of the predicate
Truth and Quantifiers

**Example:** \( D = \{ -1, 0, 1, 2 \} \)

<table>
<thead>
<tr>
<th>( P(x) )</th>
<th>True for these members of ( D )</th>
<th>True for at least one?</th>
<th>True for all?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 0 )</td>
<td>(-1)</td>
<td>\textbf{Yes}</td>
<td>\textbf{No}</td>
</tr>
<tr>
<td>( x^2 &lt; x )</td>
<td>\textit{none}</td>
<td>\textbf{No}</td>
<td>\textbf{No}</td>
</tr>
<tr>
<td>( x^2 \geq x )</td>
<td>\textit{all}</td>
<td>\textbf{Yes}</td>
<td>\textbf{Yes}</td>
</tr>
</tbody>
</table>

**Examples of statements with quantifiers**

- For every \( k \) that is a member of the set \( A = \{ 1, 2, 3, 4, 5 \} \), it is true that \( k < 20 \)
- There exists a member \( m \) of the set \( G = \{ -1, 0, 1 \} \) such that \( m^2 = m \)

**Quantifier notation**

- \( \in \): “in” or “belonging to” (set membership)  
  - \( \textbf{ex:} -1 \in \mathcal{D} \)
- \( \forall \): “for all” or “for every”
- \( \exists \): “there is (at least one)” or “there exists (at least one)”

**Rewrite the prior statements using mathematical notation**

- \( \forall k \in A, \ k < 20 \)
- \( \exists m \in G, \ m^2 = m \)
Quantified Predicates

Definitions

Quantified predicate: A predicate with one or more quantifiers
Counterexample: Example showing that a “for all” statement is false

Example 1: Translate from English to math and assess truth, for $D = \{3, 4, 5, 10, 20, 25\}$

- For every $n$ that is a member of $D$, $n < 20$:
  \[ \forall n \in D, n < 20 \text{ [False: 25 is a counterexample]} \]

- For all $n$ in the set $D$, $n < 5$ or $n$ is a multiple of 5:
  \[ \forall n \in D, (n < 5) \lor (n \text{ is a multiple of } 5) \text{ [True]} \]

- There is (at least one) $k$ in the set $D$ such that $k^2$ is also in $D$:
  \[ \exists k \in D, k^2 \in D \text{ [True, } k = 5\text{]} \]

- There exists $m$ a member of the set $D$ such that $m \geq 3$:
  \[ \exists m \in D, m \geq 3 \text{ [True, } k = 3\text{]} \]
Quantified Predicates

Definitions

Quantified predicate: A predicate with one or more quantifiers
Counterexample: Example showing that a “for all” statement is false

Example 2: Translate from math to English and assess truth, for $D = \{-2, -1, 0, 1, 2\}$

- $\forall n \in D, n > -2$: For all $n$ in the set $D$, $n > -2$ [False: $n = -2$ is a counterexample]
- $\exists n \in D, n > -2$: For at least one $n$ in the set $D$, $n > -2$ [True]
- $\forall n \in D, (n > -3) \land (n < 3)$: For all $n$ in the set $D$, $n > -3$ and $n < 3$ (i.e., $-3 < n < 3$) [True]
- $\exists m \in D, m > 10$: There exists at least one $m \in D$ such that $m > 10$ [False]
Specify the Domain!

For each quantified statement, determine the domain $D$ and rewrite formally. If the domain is ambiguous, give examples of how different domains can change the truth of the statement.

- $\mathbb{R} =$ the real numbers ($\mathbb{R}_{>0} =$ positive real numbers)
- $\mathbb{Z} =$ the integers, i.e., $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$

1. For all $x$, $x^2 \geq x$
   - If $D = \mathbb{R}$: $\forall x \in \mathbb{R}, x^2 \geq x$ [false since $x = 0.5$ is a counterexample]
   - If $D = \mathbb{Z}$: $\forall x \in \mathbb{Z}, x^2 \geq x$ [true]

2. $\forall$ even integer $m$, $m$ ends in the digit 0, 2, 4, 6, or 8
   - $D =$ set of even integers: $\forall m \in D$, $m$ ends in the digit 0, 2, 4, 6, or 8

3. There is an integer $n$ whose square root is also an integer
   - $D = \mathbb{Z}$: $\exists k \in \mathbb{Z}, \sqrt{k} \in \mathbb{Z}$

4. Every real number greater than 0 has a square that is greater than 0
   - $D = \mathbb{R}_{>0}$: $\forall n \in \mathbb{R}_{>0}, n^2 > 0$
Example: For $D = \{-2, -1, 0, 1, 2\}$, explain why each predicate is false. Write the negation in English and formally.

1. $\forall d \in D, d < -2$:
   
   There exists $d \in D$ such that $d \geq -2$
   
   $\exists d \in D, d \geq -2$

2. $\exists m \in D, m > 10$:
   
   Every element of $D$ is $\leq 10$
   
   $\forall m \in D, m \leq 10$
Negating Quantified Statements in General

Proposition

1. The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$
2. The negation of $\exists x \in D, Q(x)$ is $\forall x \in D, \neg Q(x)$
Negating Quantified Statements in General

**Proposition**

1. The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$
2. The negation of $\exists x \in D, Q(x)$ is $\forall x \in D, \neg Q(x)$

**Example:** For $D = \{-1, 0, 1, 2\}$, write the negation & determine which version is true

1. $\forall x \in D, (x \leq 0) \lor (x \geq 2)$: $\exists x \in D, \neg (x \leq 0) \land \neg (x \geq 2) \equiv \exists x \in D, (x > 0) \land (x < 2)$
2. $\exists x \in D, (x < 0) \lor (x^2 > 0)$: $\forall x \in D, (x \geq 0) \land (x^2 \leq 0)$
3. $\forall x \in D, x^2 < x$: $\exists x \in D, \neg (x^2 < x) \equiv \exists x \in D, x^2 \geq x$
4. There exists $x \in D$ such that $x^2 < x$: $\forall x \in D, x^2 \geq x$
Reminder: Predicates can have multiple arguments

Example: $P(x, y) = (x \in \mathbb{Z}) \land (y \in \mathbb{Z}) \land (x \cdot y = 36)$

▶ Evaluate: $P(9, 4) = \top$  $P(-6, -6) = \top$  $P(4, -1) = \bot$

▶ If we replace $\mathbb{Z}$ by $\mathbb{R}$, then $P(x, y) = T$ for infinitely many $(x, y)$ pairs (e.g., $x = 72, y = 0.5$)
Reminder: Predicates can have multiple arguments

Example: \( P(x, y) = (x \in \mathbb{Z}) \land (y \in \mathbb{Z}) \land (x \cdot y = 36) \)

- Evaluate: \( P(9, 4) = T \quad P(-6, -6) = T \quad P(4, -1) = F \)
- If we replace \( \mathbb{Z} \) by \( \mathbb{R} \), then \( P(x, y) = T \) for infinitely many \((x, y)\) pairs (e.g., \( x = 72, y = 0.5 \))

Multiple quantifiers of the same type (the easier case)

- There exist integers \( x \) and \( y \) such that \( x \cdot y = 36 \)
  - \( \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y = 36 \) or \( \exists x, y \in \mathbb{Z}, x \cdot y = 36 \)
- For all integers \( x \) and \( y \), it is true that \( x \cdot y = 36 \)
  - \( \forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y = 36 \) or \( \forall x, y \in \mathbb{Z}, x \cdot y = 36 \)
Mixed Quantifiers

For two variables: Two basic kinds (the truth game)

- $\forall x, \exists y, P(x, y)$: Opponent gives you $x$, you need to find $y$
- $\exists y, \forall x, P(x, y)$: You need to find $y$ that can handle any opponent’s $x$
Mixed Quantifiers

For two variables: Two basic kinds (the truth game)

- $\forall x, \exists y, P(x, y)$: Opponent gives you $x$, you need to find $y$
- $\exists y, \forall x, P(x, y)$: You need to find $y$ that can handle any opponent’s $x$

Versus ambiguous English sentences

- “For every problem there is a solution” vs “There is a solution for every problem” **same meaning**
- Let $P(x, y) = “x$ is a solution for problem $y”$
- $\forall y, \exists x, P(x, y)$ vs $\exists x, \forall y, P(x, y)$

For every problem, there exists a solution. There exist a solution that solves all problems.
Playing the Truth Game

Which of the following are true?

1. \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3 \)  
   \[ 2 + 2y = 3 \implies 2y = 1 \implies y \notin \mathbb{Z} \]

2. \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 15 \)  
   \( \text{Choose } y = 15 - x \)

3. \( \exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 15 \)  
   \( \text{You can't possibly win because you go first} \)
Negating Multiple Quantifiers

Apply our proposition from left to right:

- The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$
- The negation of $\exists x \in D, Q(x)$ is $\forall x \in D, \neg Q(x)$

Example 1


\[(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3)\] initial negation

\[\exists x \in \mathbb{Z}, \neg (\exists y \in \mathbb{Z}, x + 2y = 3)\] by proposition

\[\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg (x + 2y = 3)\] by proposition

\[\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + 2y \neq 3)\] equivalent form of “not equal”
Negating Multiple Quantifiers

Apply our proposition from left to right:

- The negation of $\forall x \in D, P(x)$ is $\exists x \in D, \neg P(x)$
- The negation of $\exists x \in D, Q(x)$ is $\forall x \in D, \neg Q(x)$

Example 1

$\neg (\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3)$ initial negation

$\exists x \in \mathbb{Z}, \neg (\exists y \in \mathbb{Z}, x + 2y = 3)$ by proposition

$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg (x + 2y = 3)$ by proposition

$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + 2y \neq 3)$ equivalent form of “not equal”

Example 2

$\neg (\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x + y = 13) \land (x \cdot y = 36))$ initial negation

$\forall x \in \mathbb{Z}, \neg (\exists y \in \mathbb{Z}, (x + y = 13) \land (x \cdot y = 36))$ by proposition

$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg ((x + y = 13) \land (x \cdot y = 36))$ by proposition

$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg (x + y = 13) \lor \neg (x \cdot y = 36)$ DeMorgan’s laws

$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + y \neq 13) \lor (x \cdot y \neq 36)$ equivalent form of “not equal”
Negating Multiple Quantifiers: Examples

Negate each quantified predicate. Which is true, the predicate or its negation?

- \( \forall x \in \mathbb{R}_{>0}, \exists y \in \mathbb{R}, (y > x) \land (x + y = 2x) \):

  \( \exists x \in \mathbb{R}_{>0}, \forall y \in \mathbb{R}, (y = x) \lor (x + y \neq 2x) \) (if true \( y = x \) but then \( y > x \) is false. So \( \lor \) is false)

- Mistake in text

- \( \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y \leq 0 \):

  \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y > 0 \)

  \( \forall x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 \geq 0 \quad \square \)

  \( \exists x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 < 0 \)