Predicates

Reading: EC 1.4

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Simple Predicates

Definition

A predicate $P(x)$ is a statement having a variable x such that whenever x is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write $P(x_1, x_2, \ldots)$.

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Example 1: $P(n) =$ "n is even"

- $n = 2$: $P(2) =$ "2 is even" $[P(2) = T]$
- $n = 17$: $P(17) =$ "17 is even" $[P(17) = F]$

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Example 2: Evaluate the following predicate for

$x = 2, 23, -5, 15$

- $R(x) =$ "$(x > 5) \wedge (x < 20)$": $R(2) = F$ $R(23) = F$

$R(-5) = F$ $R(15) = T$

Negation of Simple Predicates

Techniques carry over from negation of propositions

Example:

$P(x)$	$\neg P(x)$	
$x > 5$	$\neg(x > 5) \equiv x \leq 5$	Equivalent for all values of x
$(x > 0) \wedge (x < 10)$	$(x \leq 0) \vee (x \geq 10)$	DeMorgan's Law
$\neg(x = 8)$	$x = 8$	Double negation $\neg(\neg(x=8))$

Example 2: $P(x, y) = (x \geq 0) \vee (y \geq 0)$

- Negate $P(x, y)$: $(x<0) \wedge (y<0)$ DeMorgan's Law again
- Evaluate $P(1, 2) =$ T $P(-1, 3) =$ T $P(-7, -2) =$ F

Predicates and Sets

Informal Definition

A set is a collection of objects, which are called elements or members.

Example: $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

- For each predicate, list elements that make it true, and similarly for the negation

$P(x)$	True for ...	$\neg P(x)$	True for ...
$x \geq 8$	$8, 9, 10$	$x < 8$	$1, 2, 3, 4, 5, 6, 7$
$(x > 5) \wedge (x \text{ is even})$	$6, 8, 10$	$(x \leq 5) \vee (x \text{ is odd})$	$1, 2, 3, 4, 5, 7, 9$
$x^2 = x$	1	$x^2 \neq x$	2, 3, 4, 5, 6, 7, 8, 9, 10
$(x + 1)$ is divisible by 3	2, 5, 8	$(x+1)$ not div. by 3	1, 3, 4, 6, 7, 9, 10
$x > 0$	all	$x \leq 0$	none
$x > x^2$	none	$x \leq x^2$	all

We call D the domain of the predicate

Truth and Quantifiers

Example: $D = \{-1, 0, 1, 2\}$

$P(x)$	True for these members of D	True for at least one?	True for all?
$x < 0$	-1	Yes	No
$x^2 < x$	None	No	No
$x^2 \geq x$	All	Yes	Yes

Examples of statements with quantifiers

- For every k that is a member of the set $A = \{1, 2, 3, 4, 5\}$, it is true that $k < 20$
- There exists a member m of the set $G = \{-1, 0, 1\}$ such that $m^2 = m$

Quantifier notation

- $\in$: "in" or "belonging to" (set membership) ex: $-1 \in D$
- $\forall$: "for all" or "for every"
- $\exists$: "there is (at least one)" or "there exists (at least one)"

Rewrite the prior statements using mathematical notation

- $\forall k \in A,\ k < 20$
- $\exists m \in G,\ m^2 = m$

Quantified Predicates

Definitions

Quantified predicate: A predicate with one or more quantifiers
Counterexample: Example showing that a "for all" statement is false

Example 1: Translate from English to math and assess truth,
for $D = \{3, 4, 5, 10, 20, 25\}$

- For every n that is a member of D, $n < 20$:
  $\forall n \in D, n < 20$ [False: 25 is a counterexample]
- For all n in the set D, $n < 5$ or n is a multiple of 5:
  $\forall n \in D, (n < 5) \vee (n \text{ is a multiple of } 5)$ [True]
- There is (at least one) k in the set D such that k^2 is also in D:
  $\exists k \in D, k^2 \in D$ [True, $k = 5$]
- There exists m a member of the set D such that $m \geq 3$:
  $\exists m \in D, m \geq 3$ [True $k = 3$]

Quantified Predicates

Definitions

Quantified predicate: A predicate with one or more quantifiers
Counterexample: Example showing that a "for all" statement is false

Example 2: Translate from math to English and assess truth, for $D = \{-2, -1, 0, 1, 2\}$

- $\forall n \in D,\ n > -2$:
  For all n in the set D, $n > -2$ [False: $n = -2$ is a counterexample]
- $\exists n \in D,\ n > -2$:
  For at least one n in the set D, $n > -2$ [True]
- $\forall n \in D,\ (n > -3) \wedge (n < 3)$:
  For all n in the set D, $n > -3$ and $n < 3$ (i.e., $-3 < n < 3$) [True]
- $\exists m \in D,\ m > 10$:
  There exists at least one $m \in D$ such that $m > 10$ [False]

Specify the Domain!

For each quantified statement, determine the domain D and rewrite formally. If the domain is ambiguous, give examples of how different domains can change the truth of the statement.

- $\mathbb{R}$ = the real numbers ($\mathbb{R}_{>0}$ = positive real numbers)
- $\mathbb{Z}$ = the integers, i.e., $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$

1. For all x, $x^2 \geq x$
   - If $D = \mathbb{R}$: $\forall x \in \mathbb{R},\ x^2 \geq x$ [false since $x = 0.5$ is a counterexample]
   - If $D = \mathbb{Z}$: $\forall x \in \mathbb{Z},\ x^2 \geq x$ [true]
2. $\forall$ even integer m, m ends in the digit 0, 2, 4, 6, or 8
   - D = set of even integers: $\forall m \in D$, m ends in the digit 0, 2, 4, 6, or 8
3. There is an integer n whose square root is also an integer
   - $D = \mathbb{Z}$: $\exists k \in \mathbb{Z},\ \sqrt{k} \in \mathbb{Z}$
4. Every real number greater than 0 has a square that is greater than 0
   - $D = \mathbb{R}_{>0}$: $\forall n \in \mathbb{R}_{>0},\ n^2 > 0$

Negating Quantified Statements: Example

Example: For $D = \{-2, -1, 0, 1, 2\}$, explain why each predicate is false. Write the negation in English and formally.

1. $\forall d \in D,\ d < -2$:

   There exists $d \in D$ such that $d \geq -2$

   $\exists d \in D,\ d \geq -2$

2. $\exists m \in D,\ m > 10$:

   Every element of D is ≤ 10

   $\forall m \in D,\ m \leq 10$

Negating Quantified Statements in General

Proposition

1. The negation of $\forall x \in D,\ P(x)$ is $\exists x \in D,\ \neg P(x)$
2. The negation of $\exists x \in D,\ Q(x)$ is $\forall x \in D,\ \neg Q(x)$

Negating Quantified Statements in General

Proposition

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Example: For $D = \{-1, 0, 1, 2\}$, write the negation & determine which version is true T

1. $\forall x \in D,\ (x \leq 0) \vee (x \geq 2)$: $\exists x \in D,\ \neg(x \leq 0) \wedge \neg(x \geq 2) \equiv \exists x \in D,\ (x > 0) \wedge (x < 2)$
2. $\exists x \in D,\ (x < 0) \vee (x^2 > 0)$: $\forall x \in D,\ (x \geq 0) \wedge (x^2 \leq 0)$ F
3. $\forall x \in D,\ x^2 < x$: $\exists x \in D,\ \neg(x^2 < x) \equiv \exists x \in D,\ x^2 \geq x$ T
4. There exists $x \in D$ such that $x^2 < x$:

$\forall x \in D,\ x^2 \geq x$ T

Multiple Quantifiers

Reminder: Predicates can have multiple arguments

Example: $P(x, y) = (x \in \mathbb{Z}) \wedge (y \in \mathbb{Z}) \wedge (x \cdot y = 36)$

- Evaluate: $P(9, 4) =$ T $\quad P(-6, -6) =$ T $\quad P(4, -1) =$ F
- If we replace $\mathbb{Z}$ by $\mathbb{R}$, then $P(x, y) = T$ for infinitely many (x, y) pairs (e.g., $x = 72$, $y = 0.5$)

Multiple Quantifiers

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Multiple quantifiers of the same type (the easier case)

- There exist integers x and y such that $x \cdot y = 36$
  - $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y = 36$ or $\exists x, y \in \mathbb{Z}, x \cdot y = 36$
- For all integers x and y, it is true that $x \cdot y = 36$
  - $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y = 36$ or $\forall x, y \in \mathbb{Z}, x \cdot y = 36$

Mixed Quantifiers

For two variables: Two basic kinds (the truth game)

- $\forall x, \exists y, P(x, y)$: Opponent gives you x, you need to find y
- $\exists y, \forall x, P(x, y)$: You need to find y that can handle any opponent's x

Mixed Quantifiers

For two variables: Two basic kinds (the truth game)

- $\forall x, \exists y, P(x,y)$: Opponent gives you x, you need to find y
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Versus ambiguous English sentences

- "For every problem there is a solution" vs "There is a solution for every problem" same meaning
- Let $P(x,y) =$ "x is a solution for problem y"
- $\forall y, \exists x, P(x,y)$ vs $\exists x, \forall y, P(x,y)$

For every problem, there exists a solution

There exist a solution that solves all problems

Playing the Truth Game

Which of the following are true?

1. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3$ F take $x=2$:

   $2+2y=3$ implies $2y=1$ implies $y \notin \mathbb{Z}$

2. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 15$ T choose $y = 15 - x$

3. $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 15$ F

   you can't possibly win because you go first

Negating Multiple Quantifiers

Apply our proposition from left to right:

- The negation of $\forall x \in D,\ P(x)$ is $\exists x \in D,\ \neg P(x)$
- The negation of $\exists x \in D,\ Q(x)$ is $\forall x \in D,\ \neg Q(x)$

Example 1

$\neg(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3)$	initial negation
$\exists x \in \mathbb{Z}, \neg(\exists y \in \mathbb{Z}, x + 2y = 3)$	by proposition
$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(x + 2y = 3)$	by proposition
$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + 2y \neq 3)$	equivalent form of "not equal"

Negating Multiple Quantifiers

Apply our proposition from left to right:

- The negation of $\forall x \in D,\ P(x)$ is $\exists x \in D,\ \neg P(x)$
- The negation of $\exists x \in D,\ Q(x)$ is $\forall x \in D,\ \neg Q(x)$

Example 1

$$
\begin{aligned}
\neg(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + 2y = 3) &\qquad \text{initial negation} \\
\exists x \in \mathbb{Z}, \neg(\exists y \in \mathbb{Z}, x + 2y = 3) &\qquad \text{by proposition} \\
\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(x + 2y = 3) &\qquad \text{by proposition} \\
\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + 2y \neq 3) &\qquad \text{equivalent form of “not equal”}
\end{aligned}
$$

Example 2

$$
\begin{aligned}
\neg\big(\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x + y = 13) \wedge (x \cdot y = 36)\big) &\qquad \text{initial negation} \\
\forall x \in \mathbb{Z}, \neg\big(\exists y \in \mathbb{Z}, (x + y = 13) \wedge (x \cdot y = 36)\big) &\qquad \text{by proposition} \\
\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg\big((x + y = 13) \wedge (x \cdot y = 36)\big) &\qquad \text{by proposition} \\
\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg(x + y = 13) \vee \neg(x \cdot y = 36) &\qquad \text{DeMorgan's laws} \\
\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x + y \neq 13) \vee (x \cdot y \neq 36) &\qquad \text{equivalent form of “not equal”}
\end{aligned}
$$

Negating Multiple Quantifiers: Examples

Negate each quantified predicate. Which is true, the predicate or its negation?

- $\forall x \in \mathbb{R}_{>0}, \exists y \in \mathbb{R}, (y > x) \wedge (x + y = 2x)$:

  if true $y = x$ but then $(y > x)$ is false so $\wedge$ is false

  $\exists x \in \mathbb{R}_{>0}, \forall y \in \mathbb{R}, (y \leq x) \vee (x + y \neq 2x)$

mistake in text

- $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \cdot y \leq 0$: T Take $x = 0$

  $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x \cdot y > 0$

- $\forall x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 \geq 0$ T

  $\exists x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 < 0$