Predicates
Reading: EC 1.4

Peter J. Haas

INFO 150
Fall Semester 2019
Predicates

Simple Predicates and Their Negations
Predicates and Sets
Quantified Predicates
Negating Quantified Predicates
Multiple Quantifiers and Their Negation
Simple Predicates

Definition

A predicate $P(x)$ is a statement having a variable $x$ such that whenever $x$ is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write $P(x_1, x_2, \ldots)$. 

Example 1:

$P(n) = \text{"n is even"}$

- $P(2) = \text{"2 is even"}$ \[ P(2) = T \]
- $P(17) = \text{"17 is even"}$ \[ P(17) = F \]

Example 2:

Evaluate the following predicate for $x=2, 23, 5, 15$:

$I_R(x) = \text{"(x > 5)^{\land} (x < 20)"}$
Simple Predicates

Definition

A **predicate** \( P(x) \) is a statement having a variable \( x \) such that whenever \( x \) is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write \( P(x_1, x_2, \ldots) \).

**Example 1:** \( P(n) = \text{“} n \text{ is even} \text{”} \)
- \( n = 2: \ P(2) = \text{“} 2 \text{ is even} \text{”} \ [P(2) = T] \)
- \( n = 17: \ P(17) = \text{“} 17 \text{ is even} \text{”} \ [P(17) = F] \)
Simple Predicates

Definition

A predicate $P(x)$ is a statement having a variable $x$ such that whenever $x$ is replaced by a value, the resulting proposition is unambiguously true or false. For multiple variables, we write $P(x_1, x_2, \ldots)$.

Example 1: $P(n) = \text{“}n \text{ is even}\text{”}$

- $n = 2$: $P(2) = \text{“}2 \text{ is even}\text{”}$ [$P(2) = T$]
- $n = 17$: $P(17) = \text{“}17 \text{ is even}\text{”}$ [$P(17) = F$]

Example 2: Evaluate the following predicate for $x = 2, 23, -5, 15$

- $R(x) = \text{“}(x > 5) \land (x < 20)\text{”}$: $R(2) = F$  $R(23) = T$  $R(-5) = F$  $R(15) = T$
Negation of Simple Predicates

Techniques carry over from negation of propositions

Example:

<table>
<thead>
<tr>
<th>$P(x)$</th>
<th>$\neg P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &gt; 5$</td>
<td>$\neg(x &gt; 5) \equiv x \leq 5$</td>
</tr>
<tr>
<td>$(x &gt; 0) \land (x &lt; 10)$</td>
<td>$(x \leq 0) \lor (x \geq 10)$</td>
</tr>
<tr>
<td>$\neg(x = 8)$</td>
<td>$x = 8$</td>
</tr>
</tbody>
</table>

Equivalent for all values of $x$

Example 2: $P(x, y) = (x \geq 0) \lor (y \geq 0)$

- Negate $P(x, y)$: $(x < 0) \land (y < 0)$
- Evaluate $P(1, 2) = T$ $P(-1, 3) = \overline{T}$ $P(-7, -2) = F$
Informal Definition

A set is a collection of objects, which are called elements or members.

Example: \( D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)

- For each predicate, list elements that make it true, and similarly for the negation.

<table>
<thead>
<tr>
<th>( P(x) )</th>
<th>True for ...</th>
<th>( \neg P(x) )</th>
<th>True for ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \geq 8 )</td>
<td>8, 9, 10</td>
<td>( x &lt; 8 )</td>
<td>1, 2, 3, 4, 5, 6, 7</td>
</tr>
<tr>
<td>( (x &gt; 5) \land (x \text{ is even}) )</td>
<td>6, 8, 10</td>
<td>( (x \leq 5) \lor (x \text{ is odd}) )</td>
<td>1, 2, 3, 4, 5, 7, 9</td>
</tr>
<tr>
<td>( x^2 = x )</td>
<td>1</td>
<td>( x \neq x )</td>
<td>2, 3, 4, 5, 6, 7, 8, 9, 10</td>
</tr>
<tr>
<td>( (x + 1) \text{ is divisible by 3} )</td>
<td>2, 5, 8</td>
<td>( (x+1) \text{ not div. by 3} )</td>
<td>1, 3, 4, 6, 7, 9, 10</td>
</tr>
<tr>
<td>( x &gt; 0 )</td>
<td>all</td>
<td>( x \leq 0 )</td>
<td>none</td>
</tr>
<tr>
<td>( x &gt; x^2 )</td>
<td>none</td>
<td>( x \leq x^2 )</td>
<td>all</td>
</tr>
</tbody>
</table>

We call \( D \) the domain of the predicate.
Truth and Quantifiers

Example: \( D = \{-1, 0, 1, 2\} \)

<table>
<thead>
<tr>
<th>( P(x) )</th>
<th>True for these members of ( D )</th>
<th>True for at least one?</th>
<th>True for all?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 0 )</td>
<td>-1</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>( x^2 &lt; x )</td>
<td>None</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>( x^2 \geq x )</td>
<td>All</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Examples of statements with quantifiers

- For every \( k \) that is a member of the set \( A = \{1, 2, 3, 4, 5\} \), it is true that \( k < 20 \)
- There exists a member \( m \) of the set \( G = \{-1, 0, 1\} \) such that \( m^2 = m \)

Quantifier notation

- \( \in \): “in” or “belonging to” (set membership)
- \( \forall \): “for all” or “for every”
- \( \exists \): “there is (at least one)” or “there exists (at least one)”

Rewrite the prior statements using mathematical notation

- \( \forall k \in A, \ k < 20 \)
- \( \exists m \in G, \ m^2 = m \)
Quantified Predicates

Definitions

Quantified predicate: A predicate with one or more quantifiers
Counterexample: Example showing that a “for all” statement is false

Example 1: Translate from English to math and assess truth, for \( D = \{3, 4, 5, 10, 20, 25\}\)

- For every \( n \) that is a member of \( D \), \( n < 20 \):
  \[
  \forall n \in D, \  n < 20 \quad \text{[False: 25 is a counterexample]}
  \]

- For all \( n \) in the set \( D \), \( n < 5 \) or \( n \) is a multiple of 5:
  \[
  \forall n \in D, \ (n < 5) \lor (n \text{ is a multiple of 5}) \quad \text{[True]}
  \]

- There is (at least one) \( k \) in the set \( D \) such that \( k^2 \) is also in \( D \):
  \[
  \exists k \in D, \ k^2 \in D \quad \text{[True, } k = 5]\]

- There exists \( m \) a member of the set \( D \) such that \( m \geq 3 \):
  \[
  \exists m \in D, \ m \geq 3 \quad \text{[True, } k = 3]\]
Quantified Predicates

Definitions

Quantified predicate: A predicate with one or more quantifiers
Counterexample: Example showing that a “for all” statement is false

Example 2: Translate from math to English and assess truth, for \( D = \{-2, -1, 0, 1, 2\} \)

\[ \forall n \in D, \ n > -2: \Rightarrow \text{For all } n \text{ in the set } D, \ n > -2 \ [\text{False: } n = -2 \text{ is a counterexample}] \]

\[ \exists n \in D, \ n > -2: \Rightarrow \text{For at least one } n \text{ in the set } D, \ n > -2 \ [\text{True}] \]

\[ \forall n \in D, \ (n > -3) \land (n < 3): \Rightarrow \text{For all } n \text{ in the set } D, \ n > -3 \text{ and } n < 3 \ (\text{i.e., } -3 < n < 3) \ [\text{True}] \]

\[ \exists m \in D, \ m > 10: \Rightarrow \text{There exists at least one } m \text{ in } D \text{ such that } m > 10 \ [\text{False}] \]
Specify the Domain!

For each quantified statement, determine the domain $D$ and rewrite formally. If the domain is ambiguous, give examples of how different domains can change the truth of the statement.

- $\mathbb{R}$ = the real numbers ($\mathbb{R}_{>0}$ = positive real numbers)
- $\mathbb{Z}$ = the integers, i.e., \{0, ±1, ±2, ±3, ...\}

1. For all $x$, $x^2 \geq x$
   - If $D = \mathbb{R}$: $\forall x \in \mathbb{R}, x^2 \geq x$ [false since $x = 0.5$ is a counterexample]
   - If $D = \mathbb{Z}$: $\forall x \in \mathbb{Z}, x^2 \geq x$ [true]

2. $\forall$ even integer $m$, $m$ ends in the digit 0, 2, 4, 6, or 8
   - $D =$ set of even integers: $\forall m \in D$, $m$ ends in the digit 0, 2, 4, 6, or 8

3. There is an integer $n$ whose square root is also an integer
   - $D = \mathbb{Z}$: $\exists k \in \mathbb{Z}, \sqrt{k} \in \mathbb{Z}$

4. Every real number greater than 0 has a square that is greater than 0
   - $D = \mathbb{R}_{>0}$: $\forall n \in \mathbb{R}_{>0}, n^2 > 0$