Truth Tellers, Liars, and Propositional Logic

Reading: EC 1.3

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Smullyan's Island

You meet two inhabitants of Smullyan's Island. *A* says "exactly one of us is lying". *B* says "at least one of us is telling the truth". Who (if anyone) is telling the truth?

Strategy: Focus on the statements, not on who said them

Truth Table Analysis

Notation

- p = "A is truthful"
- q = "B is truthful"

p	q	Statement 1: Exactly one is lying	Statement 2: At least one is truthful
T	T	F	T
T	F	T	T
F	T	T	T
*F	F	F	F

Answer: Both A and B are liars

Another Smullyan's Island Example

The statements

- A: "Exactly one of use is telling the truth"
- B: "We are all lying"
- C: "The other two are lying

p	q	r	Statement 1: Exactly one truthful	Statement 2: All lying	Statement 3: A & B lying
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	F	F	F
*T	F	F	T	F	F
F	T	T	F	F	F
F	T	F	T	F	F
F	F	T	T	F	T
F	F	F	F	T	T

Answer: A **is truthful;** B **and** C **are liars**

Inconclusive or a Paradox

p	Statement 1: I am telling the truth
*T	T
*F	F

Inconclusive

p	Statement 1: I am lying
T	F
F	T

A paradox

Propositional Logic Notation

Definitions

Proposition: A sentence that is unambiguously true or false
Propositional variable: Represents a proposition (= T or F)
Formal proposition: Proposition written in formal logic notation

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Rules of formal propositions (FPs)

1. Any propositional variable is an FP
2. p and q are FPs $\Rightarrow$ $p \wedge q$ is an FP (p and q are true)
3. p and q are FPs $\Rightarrow$ $p \vee q$ is an FP (p or q or both are true)
4. p is an FP $\Rightarrow$ $\neg p$ is an FP (not p)

Example: $(p \vee q) \wedge \neg(p \vee q)$ **is a formal proposition**

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Precedence: $\neg$ **highest, then** $\wedge$**, then** $\vee$ **(like** $-$, $\times$, **and** $+$**)**

- Ex: $\neg p \wedge \neg q \vee p = \big((\neg p) \wedge (\neg q)\big) \vee p$

Logic Notation: Examples

Example 1: $p =$ **"*A* is truthful" and** $q =$ **"*B* is truthful"**

- *A* is lying: $\neg p$
- At least one of us is truthful: $p \vee q$
- Either *B* is lying or *A* is: $\neg p \vee \neg q$
- Exactly one of us is lying (exclusive or): $(\neg p \wedge q) \vee (p \wedge \neg q)$

Example 2: $e =$ **"Sue is an English major" and** $j =$ **"Sue is a Junior"**

- Sue is a Junior English major: $e \wedge j$
- Sue is either an English major or she is a Junior: $e \vee j$
- Sue is a Junior, but she is not an English major: $j \wedge \neg e$
- Sue is exactly one of the following: an English major or a Junior:

  $(e \wedge \neg j) \vee (j \wedge \neg e)$

Truth Tables for Formal Propositions

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	$\neg p$
T	F
F	T

Truth Table Examples: Complex Formulas

Example 1: $p \wedge \neg q$

p	q	$\neg q$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Example 2: $(p \vee q) \wedge \neg(p \wedge q)$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee q$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	T	F	T	F
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	F

Negation and Inequalities

Example

- $p =$ "Tammy has more than two children"
- $\neg p =$: Tammy has two or fewer children
- If $c =$ number of children, then, mathematically, $p =$ $c > 2$

Negation and Logical Equivalence

Definition

Two statements are logically equivalent if they have the same truth value for for every row of the truth table

Example: Sue is neither an English major nor a Junior

j	e	$j \vee e$	$\neg(j \vee e)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

j	e	$\neg j$	$\neg e$	$\neg j \wedge \neg e$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

← the same →

DeMorgan's Laws and Negation

Proposition (DeMorgan's Laws)

Let p and q be any propositions. Then

1. $\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$
2. $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$

Proof: Via truth tables

Example 1:

- "Sue is not both a Junior and an English major": $\neg(j \wedge e)$
- Use DeMorgan's laws to given an equivalent statement: $\neg j \vee \neg e$

Sue is not a Junior or Sue is not an English major

Example 2: "John got a B' on the test" $= (g \geq 80) \wedge (g < 90)$ [where $g =$ Johns score]

- Write the negation in math and English: $\neg((g \geq 80) \wedge (g < 90))$

$\simeq \neg(g \geq 80) \vee \neg(g < 90) = (g < 80) \vee (g \geq 90)$

John got less than B or greater than B

Tautology and Contradiction

Definition

1. A tautology is a proposition where every row of the truth table is true
2. A contradiction is a proposition where every row of the truth table is false

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg p \vee q$	$(p \vee \neg q) \vee (\neg p \vee q)$
T	T	F	F	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

p	$\neg p$	$p \wedge \neg p$
T	F	F
T	F	F
F	T	F
F	T	F

The Big Honking Theorem (BHT) of Propositions

Theorem

Let p, q and r stand for any propositions. Let t indicate a tautology and c indicate a contradiction. Then:

(a) Commutive	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
(b) Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
(c) Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(d) Identity	$p \wedge t \equiv p$	$p \vee c \equiv p$
(e) Negation	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
(f) Double negative	$\neg(\neg p) \equiv p$	
(g) Idempotent	$p \wedge p \equiv p$	$p \vee p \equiv p$
(h) DeMorgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
(i) Universal bound	$p \vee t \equiv t$	$p \wedge c \equiv c$
(j) Absorption	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
(k) Negations of t and c	$\neg t \equiv c$	$\neg c \equiv t$

Does this look familiar? similar to algebra, with $\wedge = \times$, $\vee = +$, $t = 1$, $c = 0$

(Not identical, e.g., 2^{nd} version of distributive law)

Substitution Rule: **You can replace a formula with a logically equivalent one**

Proving Logical Equivalences

Ex: Use BHT plus substitution to prove that $p \vee (\neg p \wedge q) \equiv p \vee q$

$$
\begin{aligned}
p \vee (\neg p \wedge q) &= (p \vee \neg p) \wedge (p \vee q) && \text{(c) Distributive} \\
&= t \wedge (p \vee q) && \text{(e) Negation} \\
&= (p \vee q) \wedge t && \text{(a) Commutative} \\
&= p \vee q && \text{(d) Identity}
\end{aligned}
$$