Number Puzzles and Sequences
Reading: EC 1.2

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INFO 150
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Overview and Examples

- Guess the Next Number
- Sequences and Sequence Notation
- Discovering Patterns in Sequences
- Sums
Guess the Next Number

1. 5, 7, 9, 11, 13, ?? 15
2. 1, 9, 17, 25, 33, 41, ?? 49
3. 1, 4, 9, 16, 25, 36, ?? 49
4. 2, 4, 8, 16, 32, 64, ?? 128
5. 1, 2, 6, 24, 120, 720, ?? 5040

Why do we care?

- Training for recursive thinking in a simple setting
- Used later when learning how to write proofs
- Diagnosing time and space complexity of computations
  - “At each time step, each process spawns two more processes”
  - “Each sampling step removes 2/3 of the items and adds 10 more items”
  - “The nth pass through the data has to process n rows of the table”
Guess the Next Number

1. 5, 7, 9, 11, 13, ??
2. 1, 9, 17, 25, 33, 41, ??
3. 1, 4, 9, 16, 25, 36, ??
4. 2, 4, 8, 16, 32, 64, ??
5. 1, 2, 6, 24, 120, 720, ??

Strategy: Look for Patterns

- Relate each term to previous terms (arithmetic formula)
- Describe in terms of position in sequence
- Recognize the set of integers from the examples
Patterns

Example

- Describe the sequence 1, 3, 5, 7, 9, ... each of the three ways

Solution

- Relate each term to previous terms
  - Each term is 2 more than the previous term
  - First term = 1

- Describe in terms of position in sequence
  - $n^{th}$ term = $2n - 1$

- Recognize the set of integers from the examples
  - The odd numbers (starting from 1)
More Patterns (Exercise)

Problem: For the sequence 4, 6, 8, 10, 12, ...

- Describe it each of the three ways
  (recursive, index-based, recognition-based)

  Each term is 2 more than prev. term. 1st term is 4

  $n^{th}$ term $2n+2 = 2(n+1)$

  even numbers, starting at 4

- If current term is 898, give next three terms

  900, 902, 904

- What is the 1000th term? (Which description is most helpful?)

  2002
Sequences and Sequence Notation

Recursive Formula
Each term is described in relation to previous terms via a recurrence relation.

Closed Formula
Each term is described in terms of its position in the sequence.

Sequence Notation
Sequence name is a lower-case letter (a, b, ...) and a subscript gives position in sequence: \( a_n = \text{nth term in sequence } a \)

Example
- \( a = 1, 3, 5, 7, 9, \ldots \)
- \( a_1 = 1, a_2 = 3, a_5 = 9 \)
  (it’s like a function; subscript = ordinal number)
- Closed formula: \( a_n = 2n - 1 \) (for \( n \geq 1 \))
- Recursive formula: \( a_1 = 1 \) and \( a_n = a_{n-1} + 2 \) (for \( n \geq 2 \))
Examples

For the sequence $a_n = 2^n - 1$:
- Write the first 5 terms: $a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 15, a_5 = 31$
- Value of 10th term: $a_{10} = 1023$
- Formula for $(k + 1)$st term: $2^{k+1} - 1$
- Formula for $b_i = a_{2i-3}$:

For the sequence $a_n = a_{n-1} + 5$:
- Write the first 5 terms: $a_1 = 1, a_2 = a_1 + 5 = 1 + 5 = 6, a_3 = 11, a_4 = 16$
- Recursive formula for 80th term: $a_{80} = a_{79} + 5$
- Recursive formula for $(k + 1)$st term: $a_{k+1} = a_{(k+1)-1} + 5 = a_k + 5$
- Recursive formula for $a_{2j-3}$: $a_{2j-3} = a_{(2j-3)-1} + 5 = a_{2j-4} + 5$
Discovering Patterns in Sequences

Give Recursive and closed formulas:

1. 5, 7, 9, 11, 13, ??
   \[ a_n = a_{n-1} + 2, \quad a_1 = 5 \]

2. 1, 9, 17, 25, 33, 41, ??
   \[ a_n = a_{n-1} + 8 \]

3. 1, 4, 9, 16, 25, 36, ??
   \[ a_n = a_{n-1} + 2n-1 \]

4. 2, 4, 8, 16, 32, 64, ??
   \[ a_n = 2a_{n-1}, \quad a_1 = 2 \]

5. 1, 2, 6, 24, 120, 720, ??
   \[ a_n = n! \]

(1) look for differences and quotients—how fast do the numbers grow?
(2) compare to simple series with same recurrence

See examples 1 & 2 above
A Rockstar Sequence: Fibonacci Numbers

The Sequence
1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The recurrence relation
\[ F_1 = F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3 \]

The closed formula (Binet’s formula)
\[ F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \]

Applications include (see *Fibonacci Quarterly*):
1. Fibonacci search, Fibonacci heaps
2. Biology and more (leaf/petal patterns, tree branching, ...)
Closed Formulas from Recursive Formulas (Can be Hard)

Example 1
- $a_1 = 2$ and $a_n = 3a_{n-1}$
- Compare to simpler sequence $b_n = 3^n$
- Or write out terms without simplifying
  
  \[
  a_1 = 2, \quad a_2 = 2 \cdot 3, \quad a_3 = 2 \cdot 3 \cdot 3, \ldots
  \]

Example 2
- $s_1 = 1$ and $s_n = s_{n-1} + n$
  
  \[
  \begin{array}{c|cccc}
  n & 1 & 2 & 3 & 4 \\
  \hline
  a_n & 2 & 6 & 18 & 54 \\
  3^n \cdot b_n & 3 & 9 & 27 & 81 \\
  a_n = \frac{2}{3} \cdot b_n = \frac{1}{3} 3^n & \frac{1}{3} 3^n & \frac{1}{3} 3^n & \frac{1}{3} 3^n & \frac{1}{3} 3^n \\
  \end{array}
  \]
  
  \[
  a_n = \frac{2}{3} \cdot b_n = \frac{1}{3} 3^n = 2 \cdot 2^{n-1}
  \]

\[S_n = 1 + 2 + \ldots + n\]

\[S_n \leq \frac{n(n+1)}{2}
\]
Recursive Formulas from Closed Formulas (Much Easier)

Example 1: Finding a recursive formula

- $a_n = 3n + 5$
- Compute some terms and stare at them (or try algebra)

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
</tr>
</tbody>
</table>

$a_n = a_{n-1} + 3$, $a_1 = 8$

Example 2: Verifying a recursive formula

- $a_n = 3^n - 2$
- Show that $a_n = 3a_{n-1} + 4$

\[
3a_{n-1} + 4 = 3(3^{n-1} - 2) + 4 = 3^n - 6 + 4 = 3^n - 2 = a_n
\]
Exploiting Knowledge About a Sequence

**Example 1**
- $a_1 = 11$ and $a_n = a_{n-1} + 5$
- $a_{213} = 1,071$
- What is $a_{214}$?

\[ a_{214} = a_{213} + 5 = 1,071 + 5 = 1,076 \]

**Example 2**
- $a_n = 2^n - 1$
- Sum of first 19 terms is 1,048,555
- What is the sum of the first 20 terms?
  [Hint: $2^{20} - 1 = 1,048,575$]

\[
\begin{align*}
1,048,555 & \quad + 1,048,575 \\
\hline
2,097,130 & \\
\end{align*}
\]
Sums

Notation for sums

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \text{sum of first } n \text{ terms of sequence } a \]

Extended notation for sums

\[ \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n \]

Example: Evaluate the sums

1. \[ \sum_{k=1}^{3} (2k - 1): 1 + 3 + 5 \]
2. \[ \sum_{j=0}^{4} 3^j: 1 + 3 + 9 + 27 + 81 = 121 \]
3. \[ \sum_{k=3}^{3} k^2: \quad 3^2 = 9 \]
4. \[ \sum_{k=1}^{3} \frac{1}{k(k+1)}: \quad \frac{1}{1} + \frac{1}{2.3} + \frac{1}{3.4} = \frac{1}{2} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4} \]
Sums: More Examples

Notation for sums

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \text{sum of first } n \text{ terms of sequence } a \]

Examples

- Sum of first 10 numbers in sequence \( a_k = \frac{1}{k} \) with \( k \geq 1 \)
  - \( 2 + 4 + 8 + 16 + 32 + 64 \)
  - \( 2 + 6 + 18 + 54 + 162 \)
  - \( (-4) + (-1) + 2 + 5 + 8 + 11 + 14 \)
    \[ a_k = a_{k-1} + 3 \]
Stability of Sequences

Example

Give the first 4 terms of $a_n = 3a_{n-1} - 6$ with

- $a_1 = 2$: $a_1 = 2$, $a_2 = 3a_1 - 6 = 0$, $a_3 = -6$, $a_4 = -24$
- $a_1 = 4$: $a_1 = 4$, $a_2 = 6$, $a_3 = 12$, $a_4 = 30$
- $a_1 = 3$: $a_1 = 3$, $a_2 = 3$, $a_3 = 3$, $a_4 = 3$

![Graph showing the behavior of sequences with different initial values](image-url)