Graphs

Reading: EC 7.1–7.2

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INFO 150
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Graphs

Introduction
The Königsberg Bridge Problem
Definitions and Terminology
Graph Notation for Königsberg Problem
Eulerian Graphs
Graphs with Eulerian Trails
Proofs About Graphs
You have already seen lots of graphs—we will now study their abstract properties.
Königsberg bridge problem

- Consider a walk around town: A, 6, B, 5, D, 4, A
- Eulerian trail: Crosses each bridge once
- Eulerian circuit: ... and comes back to starting point
- Does there exist an Eulerian circuit?

Euler's non-existence proof

- Suppose A is an intermediate vertex (not starting or ending)
- Bridges to A must come in pairs
- So need an even number of edges
- But every vertex has an odd number of edges
- Hence none of the four regions can be an interior vertex
- So Eulerian trail cannot exist
Origins and Euler

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Proposition 1

In any graph, if there are an odd number of edges connected to a vertex \( x \), then \( x \) cannot be an interior vertex of an Eulerian trail.

Example: Explain why the following graph does not have an Eulerian trail

Vertices \( c, f, h, j \) have odd number of edges

Question: Can we give conditions under which there exists an Eulerian trail?
Definitions and Terminology

Definitions

1. A graph $G$ consists of a set $E$ of edges and a set $V$ of vertices (also called nodes).
   - An edge is associated with one or two vertices, called endpoints.
   - Two nodes joined by an edge are called adjacent nodes.
   - An edge with one vertex is called a loop.
   - Two edges having the same endpoints are called multiple edges or parallel edges.

2. A walk is a sequence $v_1e_1v_2e_2\cdots v_ne_nv_{n+1}$ of alternating vertices and edges.
   - Each edge in the list lies between its endpoints.
   - If beginning and end vertices are the same, the walk is closed.
   - The length of a walk is the number of edges ($n$ in the above example).
   - A walk of length 0 is called a trivial walk.

3. A trail is a walk with no repeated edges; a path is a walk with no repeated vertices.
   - A circuit is a closed trail
   - A circuit having one vertex and no edges is called a trivial circuit.
   - A trail or circuit is Eulerian if it uses every edge in the graph.
   - A cycle is a nontrivial circuit in which the only repeated node is the first/last one.
Definitions: Example

1. The graph has 3 nodes and 5 edges
2. Edge 1 is a loop; edges 3 and 4 are parallel
3. Some walks in the graph:
   (a) B, 5, C, 5, B, 2, A, 2, B is a closed walk that repeats edge 5 (so not a trail)
   (b) A, 2, B, 5, C is a path (no repeated vertices)
   (c) B is a trivial walk and a trivial circuit
   (d) A, 1, A, 2, B, 5, C, 3, A is a circuit starting and ending at A
   (e) A, 2, B, 5, C, 3, A, 1, A, 4, C is an Eulerian trail
   (f) The first part of (e), A, 2, B, 5, C, 3, A, is a cycle—unlike (d), node A appears only twice

Challenge: Add one edge to the graph to create an Eulerian circuit.
Graphs in Applications

The graph that you use depends upon the application

G1: Simple Graph
G2: Directed Graph with Parallel Edges

Flight planning

- How do I get from Madrid to London? (G1 or G2)
  - Note: For G1, it suffices just to list the sequence of vertices
- What flights will take me from Madrid to London? (G2)
More Graph Notation

Definitions

1. A simple graph is a graph with no loops and no multiple (i.e., parallel) edges.
2. Notation \([a, b]\): an undirected edge with endpoints \(a\) and \(b\)
3. Notation \((a, b)\): an directed edge going from \(a\) to \(b\)

Examples

- G1 edges are: [Madrid, Paris], [London, Paris], [Madrid, Rome], [Paris, Rome]
- G2 directed edges include: (Rome, Paris), (Paris, Rome), (Rome, Madrid)
- Note: There is never any ambiguity in a simple graph
Definitions

1. An edge $e$ is incident with a node $v$ if and only if $v$ is an endpoint of $e$
2. The degree of node $v$, denoted $\text{deg}(v)$, is the number of times $v$ appears as the endpoint of an edge. (It equals the number of edges that are incident with $v$, except that loops are counted twice.)
3. A graph $G$ is connected if there is a walk between any two nodes.
4. A graph $H$ is a subgraph of a graph $G$ if all nodes and edges in $H$ are also nodes and edges in $G$.
5. A connected component of a graph $G$ is a connected subgraph $H$ of $G$ such that no other connected subgraph of $G$ containing $H$ exists.
Graph Notation for Königsberg Problem

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Example

- There is no walk from node 4 to node 6, so the graph is not connected.
- The graph has two connected components.
Königsberg Graph Notation: Example

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Example

- What is the degree of node 1?
- Find an Eulerian trail in the component on the right.
- Find an Eulerian circuit in the component on the left.
- Is the red subgraph a connected component?
Definition

A graph $G$ is Eulerian if it contains an Eulerian circuit.

Theorem 2

Let $G$ be a connected graph. The graph $G$ is Eulerian if and only if every node in $G$ has even degree.

The proof of this theorem uses induction. The basic ideas are illustrated in the next example. We reduce the problem of finding an Eulerian circuit in a big graph to finding Eulerian circuits in several smaller graphs.
How to Find an Eulerian Circuit

Given a connected graph $G$ with all nodes of even degree:

1. Find any circuit $C$ and form a graph $G'$ by removing from $G$ all edges in the circuit $C$

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2. Observe that $G'$ is not connected; call its components $H_1$ and $H_2$
3. $H_1$ and $H_2$ are smaller connected graphs, with all nodes having even degree
   (a) $H_1$ has an Eulerian circuit $C_1 = 3, 4, 5, 6, 7, 11, 12, 13, 6, 3, 11, 13, 14, 3$
   (b) $H_2$ has an Eulerian circuit $C_2 = 8, 9, 10, 8$

Graph $G$

Circuit $C = 1, 2, 3, 7, 8, 11, 14, 1$

$H_1 \rightarrow H_2$
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   (b) $H_2$ has an Eulerian circuit $C_2 = 8, 9, 10, 8$.
4. Piece together $C$, $C_1$, and $C_2$ to get the Eulerian circuit:
   - In $C$, replace 3 with $C_1$ and 8 with $C_2$:

   $1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 6, 3, 11, 13, 14, 3, 7, 8, 9, 10, 8, 11, 14, 1$.
How to Find an Eulerian Circuit

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$$1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 6, 3, 11, 13, 14, 3, 7, 8, 9, 10, 8, 11, 14, 1$$
Can you draw the figure without lifting pencil from paper or retracing a line?

- Same as asking: Can you find an Eulerian trail?
- We are *not* asking if graph is Eulerian (we’re not trying to find a circuit)
Two Simple Facts About Graphs

Theorem 3

In any graph, the sum of the degrees of the vertices is equal to twice the number of edges. I.e., \( \sum_{i=1}^{n} \deg(v_i) = 2m \), where \( v_1, v_2, \ldots, v_n \) are the vertices and \( m \) is the number of edges.

Proof:

I. Each edge has two end points (not necessarily distinct)
II. Degree of a node counts # of times it appears as endpoint of an edge
III. So when summing the degrees of nodes, we count each edge exactly twice

Corollary 4

In any graph, the number of nodes with odd degree is even.

Proof:

I. The sum of two evens is even, sum of two odds is even, sum of even and odd is odd
II. When calculating sum of all node degrees, the even numbers sum to an even number
III. By Theorem 3, all the numbers sum to an even number
IV. Therefore, the odd numbers sum to an even number
V. Therefore, there must be an even number of these
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A connected, non-Eulerian graph $G$ has an Eulerian trail if and only if $G$ has exactly two nodes of odd degree. The train must begin and end at these nodes.
General Solution for Eulerian Trails

Theorem 5

A connected, non-Eulerian graph $G$ has an Eulerian trail if and only if $G$ has exactly two nodes of odd degree. The train must begin and end at these nodes.

Proof:

- Suppose that $G$ is a connected graph that is not Eulerian but has an Eulerian trail from $v$ to $w$.
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  - Since graph is non-Eulerian, at least one node has odd degree (Theorem 2).

\[
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- Now suppose that $G$ is a connected graph with exactly two nodes, $x$ and $y$, of odd degree.
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- Now suppose that $G$ is a connected graph with exactly two nodes, $x$ and $y$, of odd degree
  - Form $G'$ by adding the edge $[x, y]$ to $G$
  - All nodes in $G'$ have even degree, so $G'$ has an Eulerian circuit (Theorem 2)
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  - Form $G'$ by adding the edge $[x, y]$ to $G$
  - All nodes in $G'$ have even degree, so $G'$ has an Eulerian circuit (Theorem 2)
  - Deleting edge $[x, y]$ from the circuit gives an Eulerian trail in $G$ from $x$ to $y$
Proofs about Graphs

We will look at two example proofs

- A direct proof
- An inductive proof of Theorem 2

We will use the techniques that we have learned, but in a new, slightly more complicated context
Proposition 4

For every connected graph $G$ with at least one edge, if $G$ has no cycles, then $G$ has at least one vertex of degree 1.
The Contrapositive Result

Proposition 4′ (Contrapositive)

For every connected graph $G$ with at least one edge, if every vertex is of degree at least 2, then $G$ has a cycle.

Proof

Let $G$ be a connected graph with at least one edge and every vertex of degree $\geq 2$. Let $n$ be the number of vertices in $G$, and choose $v_0$ to be any vertex in $G$. 

Choose vertices to build a walk:

- Since $v_0$ has degree $\geq 2$, we can pick $v_1$ to be the other endpoint of an edge incident with $v_0$.
- Since $v_1$ has degree $\geq 2$, we can pick $v_2$ to be the other endpoint of an edge incident with $v_1$ other than $[v_0, v_1]$.
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- ...and so on, until vertex $v_n$ has been chosen.

This walk uses $n+1$ vertices, but $G$ only has $n$ vertices, so there must be a first value $j$ such that $v_i = v_j$ for some $i < j$. (The "pigeonhole principle").

The cycle is then $C = v_i, v_{i+1}, v_{i+2}, ..., v_j$. 

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Lecture 15
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  - …and so on, until vertex \( v_n \) has been chosen.

- This walk uses \( n + 1 \) vertices, but \( G \) only has \( n \) vertices, so there must be a first value \( j \) such that \( v_i = v_j \) for some \( i < j \). (The “pigeonhole principle”.)

- The cycle is then \( C = v_i, v_{i+1}, v_{i+2}, \ldots, v_j \).
**Inductive Proof of Theorem 2**

**Theorem 2**

Let $G$ be a connected graph with at least one edge. The graph $G$ is Eulerian if and only if every node in $G$ has even degree.
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Lecture 15
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   - Each of these must meet $C$, say, at $v_1, v_2, \ldots, v_k$
   - “Insert” each $C_i$ into $C$ to obtain desired circuit
     (Arrange $C$ to start and end at $v_i$ and write $C_i$ in place of $v_i$ in $C$)
Reminder of Earlier Example

Graph $G$

Circuit $C = 1, 2, 3, 7, 8, 11, 14, 1$

$H_1 \rightarrow H_2$