Probability
Reading: EC 6.1–6.3

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INFO 150
Fall Semester 2019
Probability

Introduction
Definition of Probability
Rules for Computing Probabilities
Conditional Probability
Bernoulli Trials
Introduction

Probability: The rules of chance

- Most things are uncertain! (Origins of theory circa 1650, in gambling)
- Foundational to machine learning, game theory, statistical analysis, ...
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Goals

- Compute the probability of a complex event from probabilities of simpler events
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Probability: The rules of chance

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- Foundational to machine learning, game theory, statistical analysis, ...

Goals

- Compute the probability of a complex event from probabilities of simpler events
- Conquer your terrible intuition about uncertainty (see Kahneman and Tversky)
Probability Can be Non-Intuitive

Which is more likely if you pick a word at random from a dictionary: a "k" in the first position or a "k" in the third position?
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"Linda 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations". Rank the following statements in order of relative likelihood:

1. ”Linda is a bank teller”
2. ”Linda is on the executive board of UNESCO”
3. ”Linda is a bank teller and an active feminist”

Which sequence of coin flips is more likely: HHTHTT or HHHHHH?

We will learn some mathematical tools for getting the answers right
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34 people: 79.5%  28 people: 65.4%

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Basic Terminology

Setting: The outcomes of an experiment

- **Sample space** $S$: The set of possible outcomes
- Initially we’ll focus on experiments where outcomes are *equally likely*
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**Example:** Roll two dice and add up the numbers

- Attempt 1: sample space is $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ (Demo)
- Attempt 2: Sample space is
  
  $$
  S = \{ \text{double 1, double 2, double 3, double 4, double 5, double 6, double 6},
  \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\},
  \{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}$$
  
  (Demo)
- Attempt 3: ordered pair representation (see table)

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<thead>
<tr>
<th></th>
<th>Green 1</th>
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Basic Terminology

Setting: The outcomes of an experiment

- **Sample space** $S$: The set of possible outcomes
- Initially we’ll focus on experiments where outcomes are equally likely

**Example:** Roll two dice and add up the numbers

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- Attempt 2: Sample space is $S = \{\text{double } 1, \text{ double } 2, \text{ double } 3, \text{ double } 4, \text{ double } 5, \text{ double } 6\}$
  - \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\},
  - \{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\} (Demo)
- Attempt 3: ordered pair representation (see table)

We are interested in **proportion of outcomes in which some event occurs**

- Ex: Sum on two dice equals 10 (proportion of outcomes is $3/36$)
- An outcome is **successful** if the event occurs
- Technically, an event is a specified set of $E$ outcomes with $E \subseteq S$

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Definition of Probability (Equally Likely Outcomes)

Definition

Given an experiment with a sample space $S$ of equally likely outcomes and an event $E$, the probability of the event, denoted $\text{Prob}(E)$, is the ratio of the number of successful outcomes to the total number of outcomes:

$$\text{Prob}(E) = \frac{n(E)}{n(S)}$$

Example 1:
$E$ = “sum of the numbers $=$ 10"
$E = \{(4,6), (5,5), (6,4)\}$, so $\text{Prob}(E) = \frac{3}{36} = \frac{1}{12} ≈ 0.083$.

Example 2:
$E$ = “two cards drawn randomly from a 52-card deck have a value”
$S = \{(\text{AS}, 2\text{H}), (3\text{C}, \text{KD}), \ldots\}$, so $n(S) = 52 \cdot 51$.
$E = \{(\text{AS}, \text{AC}), (3\text{D}, 3\text{S}), \ldots\}$, so $n(E) = 52 \cdot 3$.
Hence $\text{Prob}(E) = \frac{52 \cdot 3}{52 \cdot 51} = \frac{1}{17}$.

Example 3:
$E$ = “last two tosses have values when tossing 5 coins"
$S = n(S) = \ldots$
$E = n(E) = \ldots$
$\text{Prob}(E) = \ldots$
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Example 2: \( E = "\text{two cards drawn randomly from a 52-card deck have same value}" \)
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Example 3: \( E = \text{“last two tosses have same value when tossing 5 coins”} \)
- \( S = \text{ordered list of length 5 with last } n \text{ elements} \)
- \( E = \text{HHHH} \)
- \( n(S) = 2^5 = 32 \)
- \( n(E) = 2^3 \cdot 2 = 16 \)
- \( \text{Prob}(E) = \frac{n(E)}{n(S)} = \frac{16}{32} = \frac{1}{2} \)
Probability of the Complement of an Event

Definition
- The Complement \( \bar{E} \) of an event \( E \) is the set of outcomes not in \( E \): \( \bar{E} = S - E \).

Complement Rule
- For any event \( E \), \( \text{Prob}(E) + \text{Prob}(\bar{E}) = 1 \).
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- For any event $E$, $\text{Prob}(E) + \text{Prob}(\bar{E}) = 1$.

**Example 1:** What is probability that same result occurs more than once in 3 die rolls?
- $E = \{(1, 6, 1), (2, 2, 4), (3, 3, 3), \ldots\}$
- $\bar{E} =$ set of rolls that are all different
- $n(\bar{E}) = P(6, 3) = 120$ and $\text{Prob}(\bar{E}) = \frac{120}{6^3} = \frac{120}{216}$
- Hence $\text{Prob}(E) = 1 - \text{Prob}(\bar{E}) = 1 - \frac{120}{216} \approx 0.44$
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**Example 2:** What is probability that, in a group of 6, at least 2 people were born in same month?

- $E = \{(1, 2, 8, 8, 8, 3), (3, 1, 4, 4, 5, 5), (2, 9, 10, 3, 4, 10) \ldots\}$
- $\bar{E}$ = set of distinct birthday-month assignments
- $n(\bar{E}) = P(12, 6)$
- Hence $\text{Prob}(E) = 1 - \text{Prob}(\bar{E}) = 1 - \frac{P(12, 6)}{12^6} = 1 - \frac{665,280}{2,985,984} \approx 0.78$
Disjoint Events

Definition

Two events $E_1$ and $E_2$ are disjoint (or mutually exclusive) if they cannot occur simultaneously in the experiment. Formally, $E_1 \cap E_2 = \emptyset$.

Examples

Dice roll:

1. $E_1 = \text{get a 3}$ and $E_2 = \text{get a 4}$ [disjoint]
2. $E_1 = \text{die is even}$ and $E_2 = \text{die is odd}$ [disjoint]
3. $E_1 = \text{3 on first roll}$ and $E_2 = \text{4 on second roll}$ [not disjoint, e.g., (3, 4)]

Problem:

Which of these pairs of events are disjoint?

1. In 4 coin tosses, $E_1 = \text{exactly 3 heads}$ and $E_2 = \text{exactly 2 tails}$
2. When choosing 4 cards, $E_1 = \text{card has a same value}$ and $E_2 = \text{card has same suit}$
3. When choosing a committee of 3 from 8 men and 12 women, $E_1 = \text{committee has a woman}$ and $E_2 = \text{committee has a man}$
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- Dice roll: $E_1 = \text{“getting a 3”}$ and $E_2 = \text{“getting a 4”}$ [disjoint]
- $E_1 = \text{“die is even”}$ and $E_2 = \text{“die is odd”}$ [disjoint]
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Problem: Which of these of these pairs of events are disjoint?

- In 4 coin tosses, $E_1 = \text{“exactly 3 heads”}$ and $E_2 = \text{“exactly 2 tails”}$

  $P(E_1) = 4/2^4 = 4/16 = 1/4$

- When choosing 4 cards, $E_1 = \text{“cards have same value”}$ and $E_2 = \text{“cards have same suit”}$

- When choosing a committee of 3 from 8 men and 12 women, $E_1 = \text{“committee has a woman”}$ and $E_2 = \text{“committee has a man”}$
The Sum Rule for Probability

Sum Rule
If $E_1$ and $E_2$ are disjoint events, then $\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2)$

Example:
For a roll of two dice, $E_1 =$ "at least one 5" and $E_2 =$ "sum equals 5"

$E_1 = \{ (1,5), (2,5), (3,5), (4,5), (6,5), (5,1), (5,2), (5,3), (5,4), (5,6) \}$, so $n(E_1) = 11$

$E_2 = \{ (1,4), (2,3), (4,1), (3,2) \}$, so $n(E_2) = 4$

Or $n(E_1) = 6$ and $n(E_2) = 4$

Hence $\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) = \frac{11}{36} + \frac{4}{36} = \frac{15}{36} = \frac{5}{12}$
The Sum Rule for Probability

**Sum Rule**

If $E_1$ and $E_2$ are disjoint events, then $\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2)$

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- $E_2 = \{(1,4), (2,3), (4,1), (3,2)\}$, so $n(E_2) = 4$
- Or $n(E_1) = 6^2 - 5^2 = 11$ and $n(E_2) = C(r - 1, r - n) = C(4,3) = 4$
- Hence $\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) = \frac{11}{36} + \frac{4}{36} = \frac{5}{12}$
The Generalized Sum Rule

If $E_1$ and $E_2$ are any events, then
\[
\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) - \text{Prob}(E_1 \text{ and } E_2)
\]

Example: When choosing 3 cards, what is probability of either 3 face cards ($E_1$) or 3 cards of same suit ($E_2$) (or both)?

- $C(52, 3)$ ways of selecting 3 cards, 13 cards per suit, 4 suits, 3 face cards per suit (jack, queen, king)

- $\text{Prob}(E_1 \text{ or } E_2) = \frac{C(12, 3)}{C(52, 3)} + 4 \cdot \frac{C(13, 3)}{C(52, 3)} - 4 \cdot 1 \frac{C(13, 3)}{C(52, 3)} = \frac{1,360}{22,100} \approx 0.0615$
Independent Events

Definition

Two events $E_1$ and $E_2$ are independent if the occurrence of one is not influenced by occurrence (or nonoccurrence) of the other.
Independent Events

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Two events \( E_1 \) and \( E_2 \) are independent if the occurrence of one is not influenced by occurrence (or nonoccurrence) of the other.

Examples

- Toss coin twice: \( E_1 = \) “heads on first toss” and \( E_2 = \) “tails on second toss” [independent]
- Choosing 2 cards from a deck: \( E_1 = \) “first card is an ace” and \( E_2 = \) “second card is an ace” [not independent]

Problem:

Which of these pairs of events are independent?

- In 4 die rolls, \( E_1 = \) “first roll sum to 7” and \( E_2 = \) “last roll sum to 10”
- When choosing a committee of 3 from 8 men and 12 women, \( E_1 = \) “committee has a woman” and \( E_2 = \) “committee has a man”
- In a household with 4 children, \( E_1 = \) “first child is male” and \( E_2 = \) “at least half the children are female”