Functions and Relations
Reading: EC 4.1–4.5

Peter J. Haas

INFO 150
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Functions and Relations
  Function Notation and Terminology
  Binary Relations
  Inverse Relations and Functions
  Composition of Functions
  Properties of Functions
  Ordering Relations
  Equivalence Relations
Notation and Terminology of Functions

Definition

A function \( f : A \rightarrow B \) associates with each input from the domain \( A \) one and only one output in the codomain \( B \) according to some rule.

Terminology

- We say that “\( f \) is a function from \( A \) to \( B \)”
- If the rule associates to element \( a \in A \) the element \( b \in B \), then we write \( f(a) = b \) and say that “\( f \) maps \( a \) to \( b \)” or “that value of \( f \) at \( a \) is \( b \)” or “\( f \) of \( a \) equals \( b \)”

Example: Define \( f : \mathbb{N} \rightarrow \mathbb{N} \) by the rule \( f(x) = 2x + 1 \)

- Q: is every element of the codomain an output of one and only one input to \( f \)?
  
  \[ \text{NO! } f(3) = 7, \quad 2x+1=0 \quad \Rightarrow \quad x = -\frac{1}{2} \quad (\text{not an integer}) \]

Example: Define \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) by the rule \( f(x) = x^2 \)

- Q: is every element of the codomain an output of one and only one input to \( f \)?
  
  \[ \text{NO! } f(-2) = f(2) = 4 \]

Functions come in many guises

- Phone directories
- Word-processing software
- Addition: \( f(3, 4) = 7 \)
- Truth tables: \( f : \{T, F\}^2 \rightarrow \{T, F\} \), e.g., \( f(p, q) = p \land q \)
- Cutting the top card: \( \kappa(\text{HCDS}) = \text{SHCD} \)
Representing a Function

An example function

- **Name:** \( f \)
- **Domain:** \( \{1, 2, 3, 4, 5\} \)
- **Codomain:** \( \mathbb{N} \)
- **Rule:** To each number in the domain, associate the square of the number

Representations of the rule

1. The above sentence
2. Algebraic formula: \( f(x) = x^2 \)
3. **Set-based description:** \( f = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\} \)
4. **Table:**

<table>
<thead>
<tr>
<th>Input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>

5. **Arrow diagram:**

Another function example:

\( S = \{a, b, c\} \)

\( f: \mathcal{P}(S) \to \{0, 1, 2, 3\} \)

\( f(A) = n(A) \)
Binary Relations

Definition

A binary relation consists of a domain $A$, a codomain $B$, and a subset of $A \times B$ called the rule for the relation.

Example: Relation $E$

- **Domain**: The set $S$ of all UMass students this semester
- **Codomain**: The set $C$ of classes offered at UMass this semester
- **Rule**: $(x, y)$ is in $E$ if student $x$ is enrolled in class $y$ this semester

Example: Relation $L$

- **Domain**: $A = \{1, 2, 3, 4\}$
- **Codomain**: $B = \{2, 3, 5\}$
- **Rule**: $L = \{(1, 2), (1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
- **Succinct representation**: $L = \{(x, y) \in A \times B : x < y\}$
- **Infix notation**: $1 \ L 2, 1 \ L 3, 2 \ L 5, 4 \ L 5, \ldots$

Observation

A function $F : A \rightarrow B$ is a special case of a relation such that for every $x \in A$, there exists exactly one element $y \in B$ for which $(x, y) \in F$
Inverse Relations

Definition

Given a relation $R$ with domain $A$ and codomain $B$, the inverse $R^{-1}$ of $R$ is the relation with domain $B$ and codomain $A$ such that

$$(x, y) \in R \text{ if and only if } (y, x) \in R^{-1}.$$ 

Example

- Relation $R$: domain $\mathbb{N}$ and codomain $\mathbb{Z}$ with rule $R = \{(x, y) \in \mathbb{N} \times \mathbb{Z} : x = y^2\}$ or equivalently $R = \{(y^2, y) : y \in \mathbb{Z}\}$
- Relation $S$: domain $\mathbb{Z}$ and codomain $\mathbb{N}$ with rule $S = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : y = x^2\}$ or equivalently $S = \{(x, x^2) : x \in \mathbb{Z}\}$
- Claim: $R$ and $S$ are inverses of each other
  1. If $(x, y) \in R$, then $x = y^2$, which means that $(y, x) = (y, y^2) \in S$ √
  2. If $(x, y) \in S$, then $x^2 = y$, which means that $(y, x) = (x^2, x) \in R$ √
Inverse Relations: More Examples

**Example 1: Relation \( E \)**

- Domain is \( A = \{1, 2, 3\} \) and codomain is \( \mathcal{P}(A) \)
- \((x, y) \in E \) (or equivalently \( x \ E \ y \)) if and only if \( x \in y \)
- \((y, x) \in E^{-1} \) (or equivalently \( y \ E^{-1} \ x \)) if and only if \( x \in y \) (also written \( y \ni x \))

![Diagram of Relation E and its inverse]

**Example:** Arrow diagram when domain and codomain are the same

\[
R = \{(A, A), (A, B), (A, C), (A, E), (C, B), (C, D), (E, A), (E, B), (E, C), (E, D)\}
\]

\[
R^{-1} = \{(A, A), (B, A), (C, A), (E, A), (B, C), (D, C), (A, E), (B, E), (C, E), (D, E)\}
\]
Inverse Functions

Definition

Functions $f : A \to B$ and $g : B \to A$ are inverses of each other if

$$f(a) = b \text{ if and only if } g(b) = a$$

for all $a \in A$ and $b \in B$. We often write $f^{-1}$ for the inverse of $f$.

Example: Prove that $f : \mathbb{Z} \to \mathbb{Z}$ with rule $f(x) = x + 3$ and $g : \mathbb{Z} \to \mathbb{Z}$ with rule $g(y) = y - 3$ are inverses of each other

- **Claim 1:** For all $a \in A$ and $b \in B$, if $f(a) = b$ then $g(b) = a$
  - Let $a, b \in \mathbb{Z}$ be given such that $f(a) = b$, i.e., $a + 3 = b$
  - Then $a = b - 3$, i.e., $g(b) = a$. ✓

- **Claim 2:** For all $a \in A$ and $b \in B$, if $g(b) = a$ then $f(a) = b$
  - Let $a, b \in \mathbb{Z}$ be given such that $g(b) = a$, i.e., $b - 3 = a$
  - Then $a + 3 = b$, i.e., $f(a) = b$. ✓

Example: for $f : \mathbb{Q} \to \mathbb{Q}$ with rule $f(x) = \frac{2}{5}x - 2$, find $f^{-1}$

- Let $a, b \in \mathbb{Q}$ be given such that $f(a) = b$, i.e., $\frac{2}{5}a - 2 = b$
- Solving for $a$, we have $a = \frac{5}{2}b + 5$
- So take $f^{-1}(y) = g(y) = \frac{5}{2}y + 5$ (can prove that $g$ is the inverse of $f$)
Inverses and Arrow Diagrams

An inverse is obtained by reversing the arrows

Example: Why is there is no function whose inverse \( g : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\} \) is given below?
Composition of Functions

**Definition**

Given \( f : A \rightarrow B \) and \( g : B \rightarrow C \), the composition \( g \circ f \) of \( g \) and \( f \) has domain \( A \), codomain \( C \) and rule \( (g \circ f)(x) = g(f(x)) \).

**Example:**

- \( f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R} \) with rule \( f(x) = \sqrt{x} \)
- \( g : \mathbb{R} \rightarrow \mathbb{R} \) with rule \( g(x) = 2x \)
- Then \( (g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = 2\sqrt{x} \)
- Then \( (f \circ g)(x) = f(g(x)) = f(2x) = \sqrt{2x} \) (what is the problem here?)

**Composition via arrow diagrams**

- Set \( \tilde{g}(x) = 2x \) with domain \( \mathbb{R}^{\geq 0} \) then \( f \circ \tilde{g} \) is well-defined
Inverse Functions Revisited

Definition

For a given set $A$, the identity function on $A$ is the function $\iota_A : A \rightarrow A$ with the rule $\iota_A(x) = x$ for all $x \in A$. We’ll often simply write $\iota$ when $A$ is clear from context. We can also write $\iota_A = \{(x, x) : x \in A\}$ when we wish to view $\iota_A$ as a binary relation.

Theorem

Functions $f : A \rightarrow B$ and $g : B \rightarrow A$ are inverses of each other if and only if $f \circ g = \iota_B$ and $g \circ f = \iota_A$.

Example 1:

- Let $f : \mathbb{Q} \rightarrow \mathbb{Q}$ be the function with rule $f(x) = \frac{2}{5}x - 2$
- Let $g : \mathbb{Q} \rightarrow \mathbb{Q}$ be the function with rule $g(x) = \frac{5}{2}x + 5$
- Then $(g \circ f)(x) = g(f(x)) = g\left(\frac{2}{5}x - 2\right) = \frac{5}{2}\left(\frac{2}{5}x - 2\right) + 5 = (x - 5) + 5 = x$
- Also, $(f \circ g)(x) = f(g(x)) = f\left(\frac{5}{2}x + 5\right) = \frac{2}{5}\left(\frac{5}{2}x + 5\right) - 2 = (x + 2) - 2 = x$

Example 2: $f : A \rightarrow A \times A$ with $f(a) = (a, a)$ and $g : A \times A \rightarrow A$ with $g(x, y) = x$

- Given $a \in A$: $(g \circ f)(a) = g(f(a)) = g(a, a) = a$, so $g \circ f = \iota_A$
- Given $(1, 2) \in A \times A$: $(f \circ g)(1, 2) = f(1) = (1, 1) \neq (1, 2)$, so $f \circ g \neq \iota_{A \times A}$