Combinatorics

Introduction
Representing Sets
Organization in Counting
Combinatorial Equivalence
Counting Lists and Permutations
Counting With Equivalence Classes: Combinations
Counting Ordered and Unordered Lists With Repetitions
What is combinatorics? Methods for answering questions about “finite structures”

- **Existence**: Is there a flight sequence that will visit 10 given cities exactly once?
- **Enumeration**: How many such sequences are there?
- **Optimization**: What is the cheapest set of such flights?

We will mostly focus on learning methods for enumeration problems

**Two key skills**

- Being able to represent objects in terms of simpler objects
- Being able to recognize when two problems are actually the same
We’ll focus on the most basic finite structure: sets

- Reminder: \{a, b, c\} is the same as \{b, c, a\} (order doesn’t matter)
- A “set” is usually a subset of a larger universe

The canonical enumeration problem

- We are given a description of a set (subset of some universe)
- We must compute the number of elements in the set

Example: How many U.S. states begin with the letter “A”?
- I.e., how many elements in the set \{Alabama, Alaska, Arizona, Arkansas\}?
- The universe is the set of all U.S. states
Example: Let $S = \{\text{Andrew, Bob, Carly, Dianne}\}$
Representing Sets

Example: Let $S = \{\text{Andrew, Bob, Carly, Dianne}\}$

1. How many ways can two prize winners be chosen from $S$?

$\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$ (unordered lists, no reps: sets)

<table>
<thead>
<tr>
<th>Can a person win both prizes?</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>Are the prizes different?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td>No</td>
<td>12</td>
<td>6</td>
</tr>
</tbody>
</table>
Representing Sets

Example: Let $S = \{\text{Andrew, Bob, Carly, Dianne}\}$

1. How many ways can two prize winners be chosen from $S$?
   
   $$\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$$
   (unordered lists, no reps: sets)

2. How many ways can we award a first prize and second prize to folks in $S$?
   
   $$\{AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC\}$$
   (ordered lists, no reps: permutations)
Representing Sets

Example: Let $S = \{\text{Andrew, Bob, Carly, Dianne}\}$

1. How many ways can two prize winners be chosen from $S$?
   \[
   \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} \quad \text{(unordered lists, no reps: sets)}
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   \[
   \{AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC\} \quad \text{(ordered lists, no reps: permutations)}
   \]

3. What if two different door prizes and same person can win both?
   \[
   \{AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC, AA, BB, CC, DD\} \quad \text{(ordered lists, reps: ordered lists)}
   \]
Representing Sets

Example: Let $S = \{\text{Andrew, Bob, Carly, Dianne}\}$

1. How many ways can two prize winners be chosen from $S$?
   $\{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$ (unordered lists, no reps: sets)

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   $\{AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC\}$
   (ordered lists, no reps: permutations)

3. What if two different door prizes and same person can win both?
   $\{AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC, AA, BB, CC, DD\}$
   (ordered lists, reps: ordered lists)

4. What if the two door prizes are identical and the same person can get both?
   $\{\{A, A\}, \{A, B\}, \{A, C\}, \{A, D\}, \{B, B\}, \{B, C\}, \{B, D\}, \{C, C\}, \{C, D\}, \{D, D\}\}$
   (unordered list, reps: bags)
   - We care about how many times each type occurs, not the order given
   - Other examples: hand of cards, bag of groceries
Representing Sets: Continued

**Definition**
- The number of \( r \)-element subsets (also called \( r \)-combinations) of the set \( \{1, 2, \ldots, n\} \) is denoted as \( C(n, r) \), also written as \( C_r^n \), \( nC_r \), and \( \binom{n}{r} \).

**Definition**
- The number of permutations of length \( r \) using elements from \( \{1, 2, \ldots, n\} \) is denoted as \( P(n, r) \), also written as \( P_r^n \) and \( nP_r \).

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<th>No</th>
</tr>
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<tbody>
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<td>Are repetitions allowed?</td>
<td>Yes</td>
<td>Ordered list</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>Permutation</td>
</tr>
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Lecture 13
Example

Which of the four structure types best characterizes the objects in each of the following situations?

- Dealing a five-card poker hand: set
- Dealing a two-card blackjack hand (one card face down and one face up): permutation
- Creating a game schedule for a sports team in baseball (can play an opponent more than once): ordered list
- Creating a game schedule for a single elimination tennis tournament: permutation
- Filling your orange plastic jack-o-lantern with halloween candy: bag

Does order matter?

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<tbody>
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<td>Ordered list</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unordered list (bag)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Permutation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Set</td>
<td></td>
<td></td>
</tr>
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</table>
Organization in counting

**Example:** How many permutations of the letters MATH are there? 24

- MATH, AMTH, AMHT, THAM, AHMT, HAMT, HMAT, MHAT, THMA, MHTA, HMTA, HATM, AHTM, MAHT, TMAH, MTHA, HTMA, TMHA, ATMH, TAHM, ATHM, TAMH, MTAH, HTAM

- Versus organizing in a table

**Example:** The number of ways to award prizes to \{Andrew, Bob, Carly, Dianne\}

- One person can win both prizes, prizes are different 16
- One person can win both prizes, both prizes the same 10
- One person cannot win both prizes, prizes are different 12
- One person cannot win both prizes, both prizes are the same 6

Elements written down in alphabetical order for convenience.
Example: Represent all outcomes of rolling a red six-sided die and a green six-sided die

36

Example: Represent all outcomes of tossing a penny, nickel, and dime together

6

Example: List all (different-looking) permutations of the four letters in the word BOOK

12

Example: List all three-element sets using letters from the work GAMES [Hint: list the set elements in alphabetical order]

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Combinatorial Equivalence

What is combinatorial equivalence?

- Informally: When two counting problems have the same answer
- Formally: When there is a one-to-one correspondence between sets

Example 1:
Why do the following questions have the same answer?
1. How many multiples of 3 are there between 100 and 300 inclusive?
2. How many integers are there between 34 and 100 inclusive?

List 1: 102 105 108 ··· 297 300

List 2: 34 35 36 ··· 99 100

Example 2:
Why do the following questions have the same answer?
1. How many ways can we distribute a red, blue, and green ball to 10 people (more than one ball per person is allowed)?
2. How many integers are there between 0 and 999 inclusive?

Distribution: Person 0 gets blue Person 2 gets blue Person 7 gets blue
Person 1 gets green Person 9 gets green Person 5 gets green

Integer: 101 229 375
Combinatorial Equivalence

What is combinatorial equivalence?

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Example 1: Why do the following questions have the same answer?

1. How many multiples of 3 are there between 100 and 300 inclusive?
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List 1: 102 105 108 ··· 297 300
List 2: 34 35 36 ··· 99 100

Example 2: Why do the following questions have the same answer?

1. How many ways can we distribute a red, blue, and green ball to 10 people (more than one ball per person is allowed)?
2. How many integers are there between 0 and 999 inclusive?

Distribution: Person 0 gets blue Person 2 gets blue Person 7 gets blue
Person 1 gets green Person 9 gets green Person 5 gets green

Integer: 101 229 375

100 - 34 + 1 = 67

Lecture 13
Combinatorial Equivalence

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List 1: 102 105 108 · · · 297 300
       ↓   ↑   ↓   · · ·   ↑   ↑
List 2: 34 35 36 · · · 99 100

Example 2: Why do the following questions have the same answer?

1. How ways can we distribute a red, blue, and green ball to 10 people (more than one ball per person is allowed)?
2. How many integers are there between 0 and 999 inclusive?

Distribution: Person 1 gets red Person 2 gets red Person 3 gets red
              Person 0 gets blue Person 2 gets blue Person 7 gets blue
              Person 1 gets green Person 9 gets green Person 5 gets green

Integer: 101 229 375

Lecture 13
Example 3: Why do the following questions have the same answer?

1. How many sets of size 2 can be made using elements from \( S = \{1, 2, 3, \ldots, 9\} \)?
2. How many sets of size 7 can be made using elements from \( S = \{1, 2, 3, \ldots, 9\} \)?

\[
\begin{align*}
T & \quad \{3, 5\} & \quad \{1, 9\} & \quad \{2, 3\} & \quad \{6, 7\} & \quad \cdots \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
S - T & \quad \{1, 2, 4, 6, 7, 8, 9\} & \quad \{2, 3, 4, 5, 6, 7, 8\} & \quad \{1, 4, 5, 6, 7, 8, 9\} & \quad \{1, 2, 3, 4, 5, 8, 9\} & \quad \cdots
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\end{align*}
\]

Example 4: Why do the following questions have the same answer?

1. How many different outcomes are there for flipping a coin five times in a row?
2. How many sets can be made using elements from \( S = \{1, 2, 3, 4, 5\} \)?

Results from coin tosses

\[
\begin{align*}
\text{THHTH} & \quad \text{HTTTT} & \quad \text{HTHTH} & \quad \cdots \\
\uparrow & \quad \uparrow & \quad \uparrow & \quad \cdots \\
\text{Subsets of } S & \quad \{2, 3, 5\} & \quad \{1\} & \quad \{2, 4\} & \quad \cdots
\end{align*}
\]

\( S \) contains positions of \( H \)'s in the sequence.
Combinatorial Equivalence, Continued

**Example 3:** Why do the following questions have the same answer?

1. How many sets of size 2 can be made using elements from $S = \{1, 2, 3, \ldots, 9\}$?
2. How many sets of size 7 can be made using elements from $S = \{1, 2, 3, \ldots, 9\}$?

\[
\begin{array}{cccccc}
T & \{3, 5\} & \{1, 9\} & \{2, 3\} & \{6, 7\} & \ldots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
S - T & \{1, 2, 4, 6, 7, 8, 9\} & \{2, 3, 4, 5, 6, 7, 8\} & \{1, 4, 5, 6, 7, 8, 9\} & \{1, 2, 3, 4, 5, 8, 9\} & \ldots \\
\end{array}
\]

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1. How many different outcomes are there for flipping a coin five times in a row?
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Results from coin tosses

\[
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\end{array}
\]

Subsets of $S$

\[
\begin{array}{cccc}
\{2, 3, 5\} & \{1\} & \{2, 4\} & \ldots \\
\end{array}
\]

**Example 5:** Why do the following questions have the same answer?

1. How many positive integer solutions are there to $x + y + z = 21$?
2. How many two-element subsets of $\{1, 2, \ldots, 20\}$ are there?

Exercise: Show that the function $f(x, y, z) = \{x, x + y\}$ is a one-to-one correspondence (one-to-one and onto) between the sets implicitly defined in 1 and 2.

Page 379 in textbook
Rules for Counting: Rule of Products

Rule of Products

If each entry in a list can be created by selecting an object from set $S_1$, then an object from $S_2$, and so on, up through selecting an object from set $S_n$, then the number of entries in the list is $n(S_1 \times S_2 \times \cdots \times S_n) = n(S_1) \times n(S_2) \times \cdots \times n(S_n)$. 

Example 1:
How many license plates with one of \{A, L, B, M\} and then four digits?

$I$ $S_1 = \{A, L, B, M\}$, $S_2 = S_3 = S_4 = S_5 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, so
$4 \times 10 \times 10 \times 10 \times 10 = 40,000$ plates

Alternatively, $S_1 = \{A, L, B, M\}$ and $S_2 = \{0000, 0001, \ldots, 9999\}$, so
$4 \times 10,000 = 40,000$ plates

Example 2:
How many numbers between 100 and 1000 have three distinct odd digits?

Three step process: choose digit from \{1, 3, 5, 7, 9\}, then choose one of the remaining 3 digits, then choose one of the remaining 4 digits, then choose one of the remaining 3 digits.

So $S_2$ depends on $S_1$, and $S_3$ depends on $S_1$ and $S_2$, but $n(S_1) \times n(S_2) \times \cdots \times n(S_n)$ are always the same.

Number of plates is $5 \times 4 \times 3 = 60$

Example 3:
How many license plates with two of \{A, L, B, M\} and then three digits?

$Lecture 13$
Rules for Counting: Rule of Products

Rule of Products

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- Number of plates is $5 \cdot 4 \cdot 3 = 60$

**Example 3:** How many license plates with two of \{A, L, B, M\} and then three digits?
- **Ex:** AA3L2, BL900
- $4 \cdot 4 \cdot 10 \cdot 10 \cdot 10 = 16,000$
Rules for Counting: Rule of Sums

**Rule of Sums**

The number of elements in two disjoint sets $S_1$ and $S_2$ is $n(S_1) + n(S_2)$.

$$n(S_1 \cup S_2) = n(S_1) + n(S_2)$$

*disjoint: $S_1 \cap S_2 = \emptyset$*
Rules for Counting: Rule of Sums

The number of elements in two disjoint sets $S_1$ and $S_2$ is $n(S_1) + n(S_2)$.

**Example 1:** How many numbers $< 1000$ comprise distinct digits from \{1, 3, 7, 9\}?

- Disjoint cases: 1-digit numbers, 2-digit numbers, and 3-digit numbers:
  \[4 + 4 \cdot 3 + 4 \cdot 3 \cdot 2 = 40\]
Rules for Counting: Rule of Sums

Rule of Sums

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  \[ 4 + 4 \cdot 3 + 4 \cdot 3 \cdot 2 = 40 \]

Example 2: How many rolls of (red, white, blue) dice have \( \geq 2 \) values the same? 
- Disjoint cases: XXY, XYX, YXX, or XXX, where X and Y are different 
- So answer is \( 6 \cdot 5 + 6 \cdot 5 + 6 \cdot 5 + 6 = 3 \cdot 6 \cdot 5 + 6 = 96 \)
Rules for Counting: Rule of Sums

Rule of Sums

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- So answer is $6 \cdot 5 + 6 \cdot 5 + 6 \cdot 5 + 6 = 3 \cdot 6 \cdot 5 + 6 = 96$

Example 3: How many rolls of (red, white, blue) dice have exactly one 3?

- Disjoint cases: 3 occurs on first, second or third roll:
- So answer is $1 \cdot 5 \cdot 5 + 5 \cdot 1 \cdot 5 + 5 \cdot 1 \cdot 5 = 3 \cdot 25 = 75$
Rules for Counting: Rule of Sums

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**Example 4:** How many 5-character license plates starting with either 1 or 2 of \( \{A, L, B, M\} \)? (and everything else a digit)

- 1 letter: \(40,000\) > from slide 12, so answers is \(40,000 + 16,000 = 56,000\)
- 2 letters: \(16,000\)
Rule of Complements

The number of elements not in set $S$ is $n(S') = n(U) - n(S)$. 

Example 1:
How many 3-letter sequences are not of the form AAA, LLL, etc.?

$I_U = \text{set of 3-letter sequences}, S = \{\text{AAA, ..., ZZZ}\}$, so $26^3 = 17,550$.

Example 2:
(a) How many 5-digit numbers use distinct digits from $\{0, 1, ..., 6\}$?

(b) How many odd? (c) How many even?

(a) Choose digits left to right (leftmost can't be 0): $6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,160$.

(b) Choose 1's digit from $\{1, 3, 5\}$, choose first digit from remaining 5 non-zeros, choose other digits from remaining 5 digits: $3 \cdot 5 \cdot 5 \cdot 4 \cdot 3 = 900$.

(c) Use rule of complements: $2,160 - 900 = 1,260$ (direct requires rule of sums).

Example 3:
How many rolls of three distinguishable dice have largest number showing being 5 or 6?
Rule of Complements

The number of elements not in set $S$ is $n(S') = n(U) - n(S)$.

**Example 1:** How many 3-letter sequences are not of the form AAA, LLL, etc.?
- $U =$ set of 3-letter sequences, $S =$ \{AAA, \ldots, ZZZ\}, so $26^3 - 26 = 17,550$
Rules for Counting: Rule of Complements

**Rule of Complements**

The number of elements not in set $S$ is $n(S') = n(U) - n(S)$.

**Example 1:** How many 3-letter sequences are not of the form AAA, LLL, etc.?
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  (c) Use rule of complements: $2,160 - 900 = 1,260$ (direct requires rule of sums)

Example 3: How many rolls of three distinguishable dice have largest number showing being 5 or 6?

$\begin{align*}
U &= \text{all rolls} \\
S &= \text{rolls } \leq 4 \\
S' &= \text{rolls } > 4 \\
\therefore n(U) &= 6 \cdot 6 \cdot 6 = 6^3 \\
n(S) &= 4^3 \\
\therefore n(U) - n(S) &= 6^3 - 4^3
\end{align*}$
Rule of Sums with Overlap

If the set of items to be counted can be broken into two overlapping sets $A$ and $B$, then the number of items is $n(A) + n(B) - n(A \cap B)$. 

Example: If we roll a die three times to make an ordered list of length 3, how many of the 6 outcomes have exactly one 1 or exactly one 6?

Let $A$ be the set of lists with exactly one 1.

Disjoint cases of $1XY$, $X1Y$, $XY1$ with $X$ and $Y$ not equal to 1.

So $n(A) = 5 \cdot 5 + 5 \cdot 5 + 5 \cdot 5 = 3 \cdot 25 = 75$.

In $B$ by same argument.

Number of lists with exactly one 1 and exactly one 6.

Choose position for 1, then position for 6, then remaining value.

$n(A \setminus B) = 3 \cdot 2 \cdot 4 = 24$.

Therefore $n(A \setminus B) + n(A) = 75 + 75 = 126$. 

Lecture 13
Rules for Counting: Rule of Sums with Overlap

If the set of items to be counted can be broken into two overlapping sets \(A\) and \(B\), then the number of items is \(n(A) + n(B) - n(A \cap B)\).

**Example:** If we roll a die three times to make an ordered list of length 3, how many of the \(6^3 = 216\) outcomes have exactly one 1 or exactly one 6?

- Let \(A\) = set of lists with exactly one 1
  - Disjoint cases of 1XY, X1Y, XY1 with X and Y not equal to 1
  - So \(n(A) = 5 \cdot 5 + 5 \cdot 5 + 5 \cdot 5 = 3 \cdot 25 = 75\)
- \(n(B) = 75\) by same argument
- Number of lists with exactly one 1 and exactly one 6
  - Choose position for 1, then position for 6, then remaining value
  - \(n(A \cap B) = 3 \cdot 2 \cdot 4 = 24\)
- Therefore \(n(A \cup B) = n(A) + n(B) - n(A \cap B) = 75 + 75 - 24 = 126\)
**Example 1:** How many numbers $\geq 200$ consist of distinct digits from $\{0, \ldots, 6\}$?

- **Algorithm 1:**
  1. Choose a ones’ digit
  2. Choose a different ten’s digit
  3. Choose a hundreds’ digit different from the other two
Your Counting Algorithm Matters!

**Example 1:** How many numbers $\geq 200$ consist of distinct digits from $\{0, \ldots, 6\}$?

- **Algorithm 1:**
  1. Choose a ones’ digit
  2. Choose a different ten’s digit
  3. Choose a hundreds’ digit different from the other two

- **Problem with product rule:** $X53$ vs $X02$

  - $2, 4, 6$, $3, 4, 5, 6$
  - Number of choices at a step depends on what you did in prior steps, so can’t just use naive product rule.
Example 1: How many numbers $\geq 200$ consist of distinct digits from $\{0, \ldots, 6\}$?

- Algorithm 1:
  1. Choose a ones' digit
  2. Choose a different ten's digit
  3. Choose a hundreds' digit different from the other two

- Problem with product rule: X53 vs X02

- Algorithm 2:
  1. Choose a hundreds' digit from $\{2, 3, 4, 5, 6\}$
  2. Choose a different ten's digit
  3. Choose a one’s digit different from the other two

- Answer $= 5 \cdot 6 \cdot 5 = 150$
Example 2: How many ways to roll 3 distinguishable die such that sum equals 10?

- **Algorithm:**
  1. Choose any element from \{1, 2, 3, 4, 5, 6\} for the first roll
  2. Choose any element from \{1, 2, 3, 4, 5, 6\} for the second roll
  3. Choose the third number by subtracting the above two numbers from 10

- Algorithm implies that answer is \(6 \cdot 6 \cdot 1 = 36\)

- **Wrong!** (Explain why)
General Formulas for Ordered Lists and Permutations

Notation

- Recall: A permutation is an ordered list with no element repeated.
- Denote by $P(n, r)$ the number of permutations from $\{1, \ldots, n\}$ of length $r$.
- Recall: $n$ factorial is $n! = n \cdot (n - 1) \cdot (n - 2) \cdots 1$ [by convention, $0! = 1$]

Theorem 1

The number of ordered lists from $\{1, \ldots, n\}$ of length $r$ is $n^r$.

Example: The number of binary sequences (e.g., 011001001) of length $r$ is $2^r$

Theorem 2

The number of permutations from $\{1, \ldots, n\}$ of length $r$ is $P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$. If $n < r$, then $P(n, r) = 0$. In the usual case where $n \geq r$, we can write $P(n, r) = \frac{n!}{(n-r)!}$.

Example: $P(5, 3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{(5)(4)(3)(2)(1)}{(2)(1)} = (5)(4)(3) = 60$
Permutations: More Examples

**Example 1:** Number of batting orders (9 players) from team of 20

\[ P(20, 9) = \frac{20!}{(20 - 9)!} = \frac{20!}{11!} = \frac{(20)(19) \cdots (12)(11)(10) \cdots 1}{(11)(10) \cdots 1} = (20)(19) \cdots (12) \approx 6 \times 10^{10} \]

**Example 2:** Number of ways to arrange 7 people in a line

\[ P(7, 7) = \frac{7!}{(7 - 7)!} = \frac{7!}{0!} = 7! = (7)(6)(5)(4)(3)(2)(1) = 5,040 \]

**Example 3:** Number of ways three married couples can stand in a movie line if spouses stand together

- Pick an order for couples to stand: can be done in \( P(3, 3) = 6 \) ways
- Pick an order for first couple to stand: can be done in \( P(2, 2) = 2 \) ways
- Pick an order for second couple to stand: can be done in \( 2 \) ways
- Pick an order for third couple to stand: can be done in \( 2 \) ways
- So total number of ways is: \( (3!)(2!)(2!)(2!) = 6 \cdot 2 \cdot 2 \cdot 2 = 48 \)
Example: How many two-element subsets of \( \{1, 2, 3, 4\} \) are there?

- Define equivalence relation on two-element permutations: “would look the same if they were sets” (1 3 equivalent to 3 1)
- Equivalence classes: \( \{1\ 3, 3\ 1\} \), \( \{4\ 2, 2\ 4\} \), \ldots
- The number of permutations of length 2 is \( P(4, 2) = 4 \cdot 3 = 12 \)
- Each equivalence class contains 2 permutations, so \( 12/2 = 6 \) equivalence classes
- Equivalence classes correspond to sets: \( \{1, 2\} \), \( \{1, 3\} \), \( \{1, 4\} \), \( \{2, 3\} \), \( \{2, 4\} \), \( \{3, 4\} \)
Combinations

Example: How many two-element subsets of \{1, 2, 3, 4\} are there?

- Define equivalence relation on two-element permutations: “would look the same if they were sets” (\{1, 3\} equivalent to \{3, 1\})
- Equivalence classes: \{1, 3, 3, 1\}, \{2, 2, 4, 4\}, ...
- The number of permutations of length 2 is \(P(4, 2) = 4 \cdot 3 = 12\)
- Each equivalence class contains 2 permutations, so \(12/2 = 6\) equivalence classes
- Equivalence classes correspond to sets: \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}
- Q: Why is the # of three-element subsets of \{1, 2, 3, 4, 5\} equal to \(P(5, 3)/6\)?
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- Q: Why is the # of three-element subsets of \( \{1, 2, 3, 4, 5\} \) equal to \( P(5, 3)/6 \)?

### Theorem 3

The number of sets of size \( r \) (called \( r \)-combinations) from \( \{1, \ldots, n\} \) is \( C(n, r) = \frac{P(n, r)}{r!} \). If \( n \geq r \), this can be written as \( C(n, r) = \frac{n!}{(r!)(n-r)!} \).
Combinations

Example: How many two-element subsets of \{1, 2, 3, 4\} are there?

- Define equivalence relation on two-element permutations: “would look the same if they were sets” (1 3 equivalent to 3 1)
- Equivalence classes: \{1 3, 3 1\}, \{4 2, 2 4\}, …
- The number of permutations of length 2 is \(P(4, 2) = 4 \cdot 3 = 12\)
- Each equivalence class contains 2 permutations, so \(12/2 = 6\) equivalence classes
- Equivalence classes correspond to sets: \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}
- Q: Why is the # of three-element subsets of \{1, 2, 3, 4, 5\} equal to \(P(5, 3)/6\) ?

Theorem 3

The number of sets of size \(r\) (called \(r\)-combinations) from \{1, \ldots, \(n\)\} is \(C(n, r) = \frac{P(n, r)}{r!}\). If \(n \geq r\), this can be written as \(C(n, r) = \frac{n!}{(r!)(n-r)!}\).

Example: How many five-person committees can be formed from the 100 member U.S. Senate? \(C(100, 5) = P(100, 5)/5! = \frac{100!}{5!95!} = \frac{(100)(99)(98)(97)(96)}{(5)(4)(3)(2)(1)} \approx 7.5 \times 10^7\)
Combinations

Example: How many two-element subsets of \( \{1, 2, 3, 4\} \) are there?

- Define equivalence relation on two-element permutations: “would look the same if they were sets” (1 3 equivalent to 3 1)
- Equivalence classes: \( \{1, 3, 3, 1\}, \{4, 2, 2, 4\}, \ldots \)
- The number of permutations of length 2 is \( P(4, 2) = 4 \cdot 3 = 12 \)
- Each equivalence class contains 2 permutations, so \( 12/2 = 6 \) equivalence classes
- Equivalence classes correspond to sets: \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \)
- Q: Why is the \( \# \) of three-element subsets of \( \{1, 2, 3, 4, 5\} \) equal to \( P(5, 3)/6 ? \)

Theorem 3

The number of sets of size \( r \) (called \( r \)-combinations) from \( \{1, \ldots , n\} \) is
\[
C(n, r) = \frac{P(n,r)}{r!}.
\]
If \( n \geq r \), this can be written as
\[
C(n, r) = \frac{n!}{(r!)(n-r)!}.
\]

Example: How many five-person committees can be formed from the 100 member U.S. Senate? \( C(100, 5) = P(100, 5)/5! = \frac{100!}{5!95!} = \frac{(100)(99)(98)(97)(96)}{(5)(4)(3)(2)(1)} \approx 7.5 \times 10^7 \)

Note: \( C(n, r) = C(n, n-r) \)

- \# of ways of choosing \( r \) items equals \# of ways of not choosing \( n-r \) items
- This can help when calculating:
  \[
  C(23, 20) = C(23, 3) = \frac{23 \cdot 22 \cdot 21}{3 \cdot 2 \cdot 1} = 1771
  \]
Examples

Example: Five-person steering committee formed from 10 women and 8 men. Of the $C(18, 5)$ possible committees, how many

1. Contain exactly three women?
   - Two-step process: Select three women, then select two men
     
     $C(10, 3) \cdot C(8, 2) = 120 \cdot 28 = 3,360$

2. Contain at least three women?
   - Disjoint cases: exactly 3 women, exactly 4 women, and exactly 5 women
     
     $C(10, 3) \cdot C(8, 2) + C(10, 4) \cdot C(8, 1) + C(10, 5) \cdot C(8, 0) = 120 \cdot 28 + 210 \cdot 8 + 252 \cdot 1 = 5,292$

3. Does not contain both Jack and Jill
   - Divide list into three disjoint parts (Jack only), (Jill only), (neither); or
   - Solve complementary problem; or
   - Divide into two parts (No Jack), (No Jill) and use sum rule with overlap

   $\begin{align*}
   &C(16, 4) + C(16, 4) + C(16, 5) = 8008 \\
   &C(14, 5) - C(16, 5) = 8008 \\
   &C(17, 5) + C(17, 5) - C(16, 5) = 8008
   \end{align*}$
More Examples

Example 1: How many ways to select a 10-person committee from 25 Democrats, 28 Republicans, 14 Independents that will have 5 Dems, 4 Repubs, and 1 Ind?

\[ C(25, 5) \cdot C(28, 4) \cdot C(14, 1) \]

Example 2: A coin is tossed 5 times, results recorded as ordered list, e.g., HTHHT

1. How many possible outcomes are there?
   \[ 2^5 = 32 \]

2. How many of these contain exactly three heads?
   - Number list positions from 1 to 5
   - Record positions that contain heads, e.g. \{1, 3, 4\}
   - Number of ways to choose three positions for heads from 5 possible positions:
     \[ C(5, 3) = C(5, 2) = \frac{5 \cdot 4}{2 \cdot 1} = 10 \]

3. Explain why \( C(5, 0) + C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5) = 2^5 \)

\( \text{Note: } C(n, 1) = n \text{ for all } n \)
Example: How many arrangements are there of 6 children holding hands in a circle?

- If children were in a line, then $6! = 720$ permutations
- Can divide into equivalence classes with 6 members each
- So number of arrangements is $720/6 = 120$
Theorem 1

The number of binary sequences of length $n$ with $r$ 0’s is $C(n, r)$.

Proof: Equivalent to choosing $r$ positions for the 0’s from $n$ possible positions.
Counting Binary Sequences and Bags

Theorem 1

The number of binary sequences of length $n$ with $r$ 0's is $C(n, r)$.

Proof: Equivalent to choosing $r$ positions for the 0's from $n$ possible positions (see above)

Theorem 2

1. The number of solutions to $x_1 + \cdots + x_n = r$ using nonnegative integers is $C(r + n - 1, r)$
2. The number of bags of $r$ items that can be made up from $n$ types of items is $C(r + n - 1, r)$

Proof:

1. Statement 2 is equivalent to Statement 1, so just prove Statement 1
2. Statement 1 is equivalent to the number of binary sequences of length $r + n - 1$ with exactly $r$ zeros
3. The number of such sequences is $C(r + n - 1, r)$ from Theorem 1

$n = 3, a + b + c = r$: $(a, b, c) \leftrightarrow 0 \ldots 0 \ 1 \ 0 \ldots 0 \ a \ b \ c$

$n = 4, r = 10$: $(1, 3, 4, 2) \leftrightarrow 0100010000100$

$n = 5, r = 12$: $(2, 0, 4, 5, 1) \leftrightarrow 001100010000010$

$c$ see above
Examples: Ordered Lists and Bags

**Example 1:** How many ordered lists of 10 letters from \{m, a, t\} have exactly 3 m’s?

- Two-step procedure: choose 3 out of 10 spaces for the m’s, then fill the remaining 7 spaces with letters from \{a, t\}
- Answer is: \( \binom{10}{3} \cdot 2^7 = 15,360 \)
Examples: Ordered Lists and Bags

Example 1: How many ordered lists of 10 letters from \( \{m, a, t\} \) have exactly 3 \( m \)'s?

- Two-step procedure: choose 3 out of 10 spaces for the \( m \)'s, then fill the remaining 7 spaces with letters from \( \{a, t\} \)
- Answer is:

Example 2: How many distinguishable arrangements of the letters in the word MISSISSIPPI are there? (11 letters long)

- Four-step procedure: choose 1 space for M, choose 4 spaces for I's, choose 4 spaces for S's, place the 2 P's in remaining spaces
- Answer is

\[
\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2}
\]

\[
= 11 \cdot 210 \cdot 15 \cdot 1 = 34,560
\]
Example 3: How many bags of 20 pieces of candy can one buy from a store having 4 types of candy?

\[ C(r + n - 1, r) = C(20 + 4 - 1, 20) = C(23, 20) = C(23, 3) = \frac{23 \cdot 22 \cdot 21}{3 \cdot 2 \cdot 1} = 1771 \]

In general: \( C(n, r) = C(n, n-r) \)
Examples, Continued

**Example 3:** How many bags of 20 pieces of candy can one buy from a store having 4 types of candy?

\[ C(r + n - 1, r) = C(20 + 4 - 1, 20) = C(23, 20) = C(23, 3) = \frac{23 \cdot 22 \cdot 21}{3 \cdot 2 \cdot 1} = 1771 \]

**Example 4:** How many bags of 10 pieces of fruit from store that carries apples, bananas, peaches, pears if we want at least one of each type?

- Two-step procedure: put one of each kind of fruit into bag, put in six more pieces of any type

- Only one way to do first step, second step is problem of putting \( r = 6 \) pieces from \( n = 4 \) types:

- Equivalent to: How many positive integer solutions to \( w + x + y + z = 10 \)?

- General formula is \( C(r - n + n - 1, r - n) = C(r - 1, r - n) \)

originally: put \( r \) pieces of fruit in bag (\( n \) types)

after step 1: put \( r - n \) pieces of fruit in bag (\( n \) types)

So, in formula, replace "\( r \)" by "\( r - n \)" everywhere
### Summary

**What?**
- Ordered lists of length \( r \)
- Permutations of length \( r \)
- Sets of size \( r \)
- Bags of size \( r \)

**How Many?**
- \( n^r \)
- \( P(n,r) \)
- \( C(n, r) \)
- \( C(r + n - 1, r) \)

### General strategies:

- Define multi-step process for creating item of interest, figure out how many ways to perform each step (product rule)

- Try to combine product rule with rule of sums (disjoint or overlapping version), i.e., break into cases

- Try to solve complementary problem