Functions and Relations
Reading: EC 4.1–4.5

Peter J. Haas

INFO 150
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Functions and Relations

- Function Notation and Terminology
- Binary Relations
- Inverse Relations and Functions
- Composition of Functions
- Properties of Functions
- Ordering Relations
- Equivalence Relations
Notation and Terminology of Functions

Definition
A function \( f : A \rightarrow B \) associates with each input from the domain \( A \) one and only one output in the codomain \( B \) according to some rule.

Terminology
- We say that “\( f \) is a function from \( A \) to \( B \)”
- If the rule associates to element \( a \in A \) the element \( b \in B \), then we write \( f(a) = b \) and say that “\( f \) maps \( a \) to \( b \)” or “that value of \( f \) at \( a \) is \( b \)” or “\( f \) of \( a \) equals \( b \)”

Example: Define \( f : \mathbb{N} \rightarrow \mathbb{N} \) by the rule \( f(x) = 2x + 1 \)
- Q: is every element of the codomain an output of one and only one input to \( f \)?
  - \( \text{NO! } f(\frac{1}{2}) = 0 \quad 2 \cdot \frac{1}{2} + 1 = 0 \quad \Rightarrow \quad \frac{1}{2} \notin \mathbb{Z} \) (not an integer)

Example: Define \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) by the rule \( f(x) = x^2 \)
- Q: is every element of the codomain an output of one and only one input to \( f \)?
  - \( \text{NO! } f(-2) = f(2) = 4 \)

Functions come in many guises
- Phone directories
- Word-processing software
- Addition: \( f(3, 4) = 7 \)
- Truth tables: \( f : \{T, F\}^2 \rightarrow \{T, F\} \), e.g., \( f(p, q) = p \land q \)
- Cutting the top card: \( \kappa(HCDS) = SHCD \)

Lecture 13
Representing a Function

An example function

- **Name**: \( f \)
- **Domain**: \( \{1, 2, 3, 4, 5\} \)
- **Codomain**: \( \mathbb{N} \)
- **Rule**: To each number in the domain, associate the square of the number

Representations of the rule

1. The above sentence
2. Algebraic formula: \( f(x) = x^2 \)
3. **Set-based description**: \( f = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\} \)
4. Table:

<table>
<thead>
<tr>
<th>Input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>

5. Arrow diagram:
Binary Relations

**Definition**

A binary relation consists of a **domain** $A$, a **codomain** $B$, and a subset of $A \times B$ called the **rule** for the relation.

**Example: Relation $E$**

- **Domain**: The set $S$ of all UMass students this semester
- **Codomain**: The set $C$ of classes offered at UMass this semester
- **Rule**: $(x, y)$ is in $E$ if student $x$ is enrolled in class $y$ this semester

**Example: Relation $L$**

- **Domain**: $A = \{1, 2, 3, 4\}$
- **Codomain**: $B = \{2, 3, 5\}$
- **Rule**: $L = \{(1, 2), (1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
- **Succinct representation**: $L = \{(x, y) \in A \times B : x < y\}$
- **Infix notation**: $1 \ L \ 2, \ 1 \ L \ 3, \ 2 \ L \ 5, \ 4 \ L \ 5, \ \ldots$

**Observation**

A function $F : A \rightarrow B$ is a **special case** of a relation such that for every $x \in A$, there exists exactly one element $y \in B$ for which $(x, y) \in F$.
Inverse Relations

Definition

Given a relation $R$ with domain $A$ and codomain $B$, the inverse $R^{-1}$ of $R$ is the relation with domain $B$ and codomain $A$ such that

$$(x, y) \in R \quad \text{if and only if} \quad (y, x) \in R^{-1}.$$  

Example

- Relation $R$: domain $\mathbb{N}$ and codomain $\mathbb{Z}$ with rule $R = \{(x, y) \in \mathbb{N} \times \mathbb{Z} : x = y^2\}$
  or equivalently $R = \{(y^2, y) : y \in \mathbb{Z}\}$

- Relation $S$: domain $\mathbb{Z}$ and codomain $\mathbb{N}$ with rule $S = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : y = x^2\}$
  or equivalently $S = \{(x, x^2) : x \in \mathbb{Z}\}$

- Claim: $R$ and $S$ are inverses of each other
  1. If $(x, y) \in R$, then $x = y^2$, which means that $(y, x) = (y, y^2) \in S$
  2. If $(x, y) \in S$, then $x^2 = y$, which means that $(y, x) = (x^2, x) \in R$
Inverse Relations: More Examples

**Example 1:** Relation $E$

- Domain is $A = \{1, 2, 3\}$ and codomain is $\mathcal{P}(A)$
- $(x, y) \in E$ (or equivalently $x \in E \; y$) if and only if $x \in y$
- $(y, x) \in E^{-1}$ (or equivalently $y \in E^{-1} \; x$) if and only if $x \in y$ (also written $y \ni x$)

![Diagram showing $E$ and $E^{-1}$]

**Example:** Arrow diagram when domain and codomain are the same

$R = \{(A, A), (A, B), (A, C), (A, E), (C, B), (C, D), (E, A), (E, B), (E, C), (E, D)\}$

$R^{-1} = \{(A, A), (B, A), (C, A), (E, A), (B, C), (D, C), (A, E), (B, E), (C, E), (D, E)\}$

![Diagram showing $R$ and $R^{-1}$]
Inverse Functions

Definition

Functions $f: A \rightarrow B$ and $g: B \rightarrow A$ are **inverses** of each other if

$$f(a) = b$$

if and only if

$$g(b) = a$$

for all $a \in A$ and $b \in B$. We often write $f^{-1}$ for the inverse of $f$.

Example: Prove that $f: \mathbb{Z} \rightarrow \mathbb{Z}$ with rule $f(x) = x + 3$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ with rule $g(y) = y - 3$ are inverses of each other

- **Claim 1:** For all $a \in A$ and $b \in B$, if $f(a) = b$ then $g(b) = a$
  - Let $a, b \in \mathbb{Z}$ be given such that $f(a) = b$, i.e., $a + 3 = b$
  - Then $a = b - 3$, i.e., $g(b) = a$. ✓

- **Claim 2:** For all $a \in A$ and $b \in B$, if $g(b) = a$ then $f(a) = b$
  - Let $a, b \in \mathbb{Z}$ be given such that $g(b) = a$, i.e., $b - 3 = a$
  - Then $a + 3 = b$, i.e., $f(a) = b$. ✓

Example: for $f: \mathbb{Q} \rightarrow \mathbb{Q}$ with rule $f(x) = \frac{2}{5}x - 2$, find $f^{-1}$

- Let $a, b \in \mathbb{Q}$ be given such that $f(a) = b$, i.e., $\frac{2}{5}a - 2 = b$
- Solving for $a$, we have $a = \frac{5}{2}b + 5$
- So take $f^{-1}(y) = g(y) = \frac{5}{2}y + 5$
Inverses and Arrow Diagrams

An inverse is obtained by reversing the arrows

Example: Why is there is no function whose inverse \( g : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\} \) is given below?

what is \( g^{-1}(3) \)?
what is \( g^{-1}(4) \)?
Composition of Functions

Definition

Given \( f : A \to B \) and \( g : B \to C \), the composition \( g \circ f \) of \( g \) and \( f \) has domain \( A \), codomain \( C \) and rule \( (g \circ f)(x) = g(f(x)) \).

Example:

- \( f : \mathbb{R}^\geq 0 \to \mathbb{R} \) with rule \( f(x) = \sqrt{x} \)
- \( g : \mathbb{R} \to \mathbb{R} \) with rule \( g(x) = 2x \)
- Then \( (g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = 2\sqrt{x} \)
- Then \( (f \circ g)(x) = f(g(x)) = f(2x) = \sqrt{2x} \) (what is the problem here?)

Composition via arrow diagrams

\[ g \circ f \]

\[ f \circ g \]
Inverse Functions Revisited

Definition

For a given set \( A \), the **identity function on** \( A \) is the function \( \iota_A : A \to A \) with the rule \( \iota_A(x) = x \) for all \( x \in A \). We’ll often simply write \( \iota \) when \( A \) is clear from context. We can also write \( \iota_A = \{(x, x) : x \in A\} \) when we wish to view \( \iota_A \) as a binary relation.

Theorem

Functions \( f : A \to B \) and \( g : B \to A \) are inverses of each other if and only if \( f \circ g = \iota_B \) and \( g \circ f = \iota_A \).

Example 1:

- Let \( f : \mathbb{Q} \to \mathbb{Q} \) be the function with rule \( f(x) = \frac{2}{5}x - 2 \)
- Let \( g : \mathbb{Q} \to \mathbb{Q} \) be the function with rule \( g(x) = \frac{5}{2}x + 5 \)
- Then \( (g \circ f)(x) = g(f(x)) = g(\frac{2}{5}x - 2) = \frac{5}{2}(\frac{2}{5}x - 2) + 5 = (x - 5) + 5 = x \)
- Also, \( (f \circ g)(x) = f(g(x)) = f(\frac{5}{2}x + 5) = \frac{2}{5}(\frac{5}{2}x + 5) - 2 = (x + 2) - 2 = x \)

Example 2: \( f : A \to A \times A \) with \( f(a) = (a, a) \) and \( g : A \times A \to A \) with \( g(x, y) = x \)

- Given \( a \in A \): \( (g \circ f)(a) = g(f(a)) = g(a, a) = a \), so \( g \circ f = \iota_A \)
- Given \((1, 2) \in A \times A\): \( (f \circ g)(1, 2) = f(1) = (1, 1) \neq (1, 2) \), so \( f \circ g \neq \iota_{A \times A} \)
Properties of Functions

**Definition**

The function \( f : A \to B \) is invertible if there is a function \( f^{-1} : B \to A \) such that \( f(x) = y \) if and only if \( f^{-1}(y) = x \). By symmetry of the definition, \((f^{-1})^{-1} = f\).

**Example**

- With \( A = B = \mathbb{R}^\geq_{0} \), if \( f \) has rule \( f(x) = x^2 \), then \( f^{-1}(x) = \sqrt{x} \).
- Arrow diagram:

```plaintext
\[ \begin{array}{cccc}
a & f & 1 \\
b & 2 \\
c & 3 \\
d & 4 \\
\end{array} \]

\[ \begin{array}{cccc}
1 & f^{-1} & a \\
2 & b \\
3 & c \\
4 & d \\
\end{array} \]

\( (f^{-1} \circ f)(x) = x \)

**A non-invertible function \( g \)**

- Problem 1: no arrow points to 4 (\( g \) is not onto)
- Problem 2: two arrows point to 3 (\( g \) is not one-to-one)

**Example:** Which functions are invertible?

- \( f : \mathbb{Z} \to \mathbb{Z} \) with \( f(x) = 2x + 3 \) **Not onto.** There exists no \( x \in \mathbb{Z} \) st. \( f(x) = 0 \).
- \( g : \mathbb{Z} \to \mathbb{N} \) with \( g(x) = \begin{cases} -2z & \text{if } z \leq 0 \\ 2z - 1 & \text{if } z > 0 \end{cases} \) **Invertible, one-to-one and onto**
  
  \( \text{Even: } g(-\frac{x}{2}) = y \), \( \text{Odd: } g(\frac{x+1}{2}) = y \)

- \( h : \mathbb{N} \to \mathbb{N} \) with \( h(n) = \text{sum of digits in the numeral } n \) **not 1-to-1**, \( f(31) = f(13) \)

Lecture 13
Proofs About Functions, Continued

Proposition 1

If \( f : A \to B \) is one-to-one and \( g : B \to C \) is one-to-one, then \( (g \circ f) : A \to C \) is one-to-one.

Proof (prove the contrapositive)

1. Write \( h = g \circ f \) and let \( x_1, x_2 \in A \) be given such that \( h(x_1) = h(x_2) \)
2. This means that \( g(f(x_1)) = g(f(x_2)) \)
3. Since \( g \) is one-to-one, this means that \( f(x_1) = f(x_2) \)
4. Since \( f \) is one-to-one, this means that \( x_1 = x_2 \)

Example: \( f : \mathbb{N} \to \mathbb{N} \) with \( f(x) = 5x + 7 \) and \( g : \mathbb{N} \to \mathbb{Q} \) with \( g(n) = \frac{5}{n+2} \)

1. Write \( h = g \circ f \) and let \( x_1, x_2 \in A \) be given such that \( h(x_1) = h(x_2) \)
2. This means that \( g(f(x_1)) = g(f(x_2)) \) or \( \frac{5}{f(x_1)+2} = \frac{5}{f(x_2)+2} \) or \( \frac{f(x_1)+2}{5} = \frac{f(x_2)+2}{5} \)
3. Multiply by 5 and subtract 2 on both sides: \( f(x_1) = f(x_2) \) or \( 5x_1 + 7 = 5x_2 + 7 \)
4. Subtract 7 and divide by 5 on both sides to get \( x_1 = x_2 \)
Proofs About Functions, Continued

Example: Prove that \( f : \mathbb{R}^+ \to (1, \infty) \) with \( f(x) = \frac{x+1}{x} \) is onto

- Strategy: pick an arbitrary \( y \) in the codomain and find an \( x \) in the domain such that \( f(x) = y \) [Use \( x = f^{-1}(y) \) if \( f^{-1} \) exists, else any \( x \) with an arrow to \( y \)]

- Proof:
  1. Let \( y \in (1, \infty) \) and set \( x = \frac{1}{y-1} \)
  2. Since \( y > 1 \), we see that \( x \in \mathbb{R}^+ \)
  3. Also, \( f(x) = f\left(\frac{1}{y-1}\right) = \frac{\frac{1}{y-1}+1}{\frac{1}{y-1}} = \frac{\frac{y-1+1}{y-1}}{\frac{1}{y-1}} = y \)

Proposition 2  \( \text{Similar proof to Proposition 1} \)

If \( f : A \to B \) is onto and \( g : B \to C \) is onto, then \( (g \circ f) : A \to C \) is onto.

Theorem

If \( f : A \to B \) is invertible and \( g : B \to C \) is invertible, then \( (g \circ f) : A \to C \) is invertible.

Proof

1. Since \( f \) and \( g \) are invertible, they are each one-to-one and onto
2. By Propositions 1 and 2, \( g \circ f \) is one-to-one and onto
3. Hence \( g \circ f \) is invertible
Properties of Relations

Definition

Let $R$ be a binary relation on a set $A$

1. $R$ is reflexive if $(a, a) \in R$ for all $a \in R$ (must have loops)
2. $R$ is antisymmetric if $(a, b) \in R$ and $a \neq b$ implies $(b, a) \notin R$ (no double arrows)
3. $R$ is transitive if $(a, b), (b, c) \in R$ implies $(a, c) \in R$ (if you can get from $a$ to $c$ following two arrows, you can also get there following one arrow)

Definition

A relation $R$ on a set $A$ is partial order if it is antisymmetric, transitive, and reflexive

Examples

1. $A = \{1, 2, 3, 4\}$: $a R_1 b$ means $a \leq b$
2. $A = \mathcal{P}(\{1, 2, 3\})$: $a R_2 b$ means $a \subseteq b$
3. $A = \{1, 2, 3, 6\}$: $a R_3 b$ means $a$ divides $b$

$R_1 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
Proofs About Properties

**Example:** Prove the reflexive property for \( R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \text{ is even}\}\)

1. Let \( a \in \mathbb{Z} \) be given
2. Since \( a - a = 0 \), which is even, we have that \((a, a) \in R\)

**Example:** Prove the transitive property for \( R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \text{ is even}\}\)

1. Let \((a, b), (b, c) \in R\) be given [We’ll prove that \((a, c) \in R\)]
2. Since \( a - b \) and \( b - c \) are even, we can write \( a - b = 2K \) and \( b - c = 2L \) for some integers \( K \) and \( L \)
3. Therefore \( a - c = (a - b) + (b - c) = 2K + 2L = 2(K + L) \)
4. Thus \((a - c)\) is even and hence \((a, c) \in R\)

**Example:** Prove the antisymmetric property for \( R = \{(s, t) \in \mathcal{P}({1, 2, 3, 4})^2 : s \subseteq t\}\)

Show: for all \( s, t \in \mathcal{P}({1, 2, 3, 4})\), if \((s, t) \in R\) and \((t, s) \in R\), then \(s = t\)

1. Let \( s, t \in \mathcal{P}({1, 2, 3, 4})\) be given
2. Since \((s, t) \in R\) and \((t, s) \in R\), we have that \(s \subseteq t\) and \(t \subseteq s\)
3. It follows that \(s = t\) by definition of set equality

\[
\text{Contrapositive} \quad (a \neq b) \rightarrow \neg \left( (a, b) \in R \land (b, a) \in R \right) \\

L(a, b) \in R \land (b, a) \in R) \rightarrow (a = b)
\]
Other Types of Orders

Definition

A relation $R$ over a set $A$ is **irreflexive** if $(a, a) \not\in R$ for all $a \in R$. A **strict partial ordering** on $A$ is a relation $R$ on $A$ that is transitive, antisymmetric, and irreflexive.

Notes

- Irreflexive means **no loops** in an arrow diagram
- A relation $R$ can be neither reflexive or irreflexive if some (but not all) nodes in the arrow diagram have loops

Example:

- Strict subset relation: write $A \subset B$ if $A \subseteq B$ and $B - A \neq \emptyset$
- Then $R = \{(A, B) \in \mathcal{P}({1, 2, 3, 4})^2 : A \subset B\}$ is a strict partial ordering

Definition

A relation $R$ on $A$ is a **total ordering** if it is a partial ordering and also satisfies the property:
For all $a, b \in A$, if $a \neq b$, then either $(a, b) \in R$ or $(b, a) \in R$. A **strict total ordering** has the same properties except that it is irreflexive.
Types of Orderings: Examples

Notation:
- $A = \{1, 2, 4, 8\}$, $B = \mathcal{P}(\{1, 2, 3\})$, $C = \{0, 1\}^4$
- $V(\alpha) =$ value of binary numeral $\alpha$, e.g., $V(0101) = 5$

Example 1: $R_1 = \{(x, y) \in A^2 : x \leq y\}$ total ordering

Example 2: $R_2 = \{(x, y) \in B^2 : x \subseteq y\}$ partial ordering (incomparable subsets)

Example 3: $R_2 = \{(S, T) \in B^2 : \text{every element in } S \text{ is } \leq \text{ every element in } T\}$ partial ordering: $\{1, 3\}$ and $\{2, 3\}$ are incomparable

Example 4: $R_2 = \{(S, T) \in B^2 : n(S) < n(T)\}$ strict partial ordering: $\{1, 3\}$, $\{1, 3\}$ incomparable

Example 5: $R_2 = \{(S, T) \in B^2 : \text{sum of elements in } S \text{ is } \leq \text{ sum of elements in } T\}$ ordering: $\{1, 2\}$, $\{1, 3\}$ incomparable

Example 6: $R_2 = \{(\alpha, \beta) \in C^2 : \alpha \text{ has fewer 1's than } \beta \text{ has}\}$ strict partial order $1100$, $0101$ incomparable

Example 7: $R_2 = \{(\alpha, \beta) \in C^2 : V(\alpha) \leq V(\beta)\}$ total order $1010$, $0101$ incomparable
Equivalence Relations

Definition

A partition of a set $A$ is a set $S = \{S_1, S_2, S_3, \ldots\}$ such that

1. For all $i$, $S_i \neq \emptyset$
2. For all $i, j$: if $S_i \neq S_j$, then $S_i \cap S_j = \emptyset$
3. $S_1 \cup S_2 \cup S_3 \cup \ldots = A$

Example 1: $A = \{1, 2, 3, 4, 5, 6\}$

- $R = \{(a, b) \in A \times A : a - b \text{ is even}\}$
- $S = \{\{1, 3, 5\}, \{2, 4, 6\}\}$

Example 2:

- $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \text{ is divisible by 4}\}$
- $S = \{P_0, P_1, P_2, P_3\}$ where $P_i = \{a \in \mathbb{Z} : a = 4k + i \text{ for some } k \in \mathbb{Z}\}$
Formal Properties of an Equivalence Relation

Definition

A relation \( R \) on \( A \) is **symmetric** if for all \( a, b \in A \), if \( (a, b) \in A \) then \( (b, a) \in A \).

Theorem

A relation \( R \) on \( A \) is an equivalence relation if and only if it is **reflexive**, **symmetric**, and **transitive**.

Example: For each relation, determine whether it is an equivalence relation

- \( T_1 = \{(a, b) \in \mathbb{Z}^2 : b - a \text{ is divisible by } 5\} \): Yes (similar to slide 16 for transitive)
- \( T_2 = \{(a, b) \in \mathbb{Z}^2 : a^2 - b^2 \text{ is divisible by } 5\} \): Yes (same argument)
- \( T_3 = \{(a, b) \in \mathbb{Z}^2 : |a - b| \leq 2\} \): No. Not transitive

\[
\begin{align*}
(1, 2) : & \quad |1 - 2| \leq 2 \\
(2, 4) : & \quad |2 - 4| \leq 2 \\
\text{but } (1, 4) \notin T_3 : & \quad |1 - 4| > 2
\end{align*}
\]