Sets
Reading: EC 3.1-3.3

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INFO 150
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Sets

Definitions
Defining Sets
Set Operations
The Inclusion-Exclusion Principle
Cartesian Products
The Power Set
Some Common Sets

**Loose definition:** A set is a collection of objects (called members or elements)

- This loose general definition can lead to paradoxes (e.g., Russell’s Paradox)
- To stay out of trouble, we will work with a small number of well-understood sets

### Basic Sets

- \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \): The set of natural numbers
- \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \): The set of integers
- \( \mathbb{Q} \): The set of rational numbers, e.g., ratios of integers such as \( \frac{2}{3}, \frac{3}{1}, \frac{-17}{4} \)
- \( \mathbb{R} \): The set of real numbers, i.e., decimal numbers with possibly infinite strings of digits after the decimal point

### Variations on Basic Sets

- \( \mathbb{R}^+ \): The set of positive real numbers
- \( \mathbb{R}^{\geq 0} \): The set of nonnegative real numbers
- \( \mathbb{Q}^+ \): The set of positive rationals
- \( \mathbb{Q}^{\geq 0} \): The set of nonnegative rationals
- \( \mathbb{Z}^+ \): The set of positive integers
- \( \mathbb{Z}^{\geq 0} \): The same as \( \mathbb{N} \)
Subsets

Definitions for Subsets

- \( x \in A \): The element \( x \) is a **member** of the set \( A \)
- \( A \subseteq B \): \( A \) is a **subset** of \( B \), i.e., every element in \( A \) is also in \( B \)
- \( A = B \): Set \( A \) and \( B \) contain exactly the same elements
- \( \emptyset \): The **empty set**, i.e., the set that contains no elements
- \( U \): For any given discussion, all sets will be subsets of a larger set called the **universal set** or the **universe**

Some Formal Definitions

- \( A \subseteq B \): \( \forall x \in U, \ (x \in A) \to (x \in B) \)
- \( A = B \): \( A \subseteq B \) and \( B \subseteq A \)

True or False? (If false, give a counterexample)

- \( \mathbb{Z} \subseteq \mathbb{N} \): False \((-3)\)
- \( \mathbb{N} \subseteq \mathbb{Z} \): True
- \( \mathbb{Q} \subseteq \mathbb{Z} \): False \(\left(\frac{3}{4}\right)\)
- \( \mathbb{Z} \subseteq \mathbb{Q} \): True
- \( \mathbb{R} \subseteq \mathbb{Q} \): False \(\sqrt{2}\)
- \( \mathbb{Q} \subseteq \mathbb{R} \): True
Digression: $\sqrt{2}$ is Irrational

Theorem

$\sqrt{2}$ is irrational

Proof by Contradiction:

1. Suppose the theorem is false
2. Then we can write $\sqrt{2} = \frac{a}{b}$ where $a$ and $b$ are relatively prime
3. So $2 = \frac{a^2}{b^2}$, or $a^2 = 2b^2$
4. Therefore $a^2$ is even, which implies that $a$ is even (see end of Lecture 6)
5. Therefore $a = 2k$ for some integer $k$
6. So $a^2 = 4k^2 = 2b^2$ and hence $b^2 = 2k^2$
7. Therefore $b^2$ is even, so that $b$ is even
8. Thus $a$ and $b$ are both even
9. This contradicts the assumption that $a$ and $b$ are relatively prime
10. Since assuming that the theorem is false leads to a contradiction, the theorem must be true.
More Examples

1. \( \{1, 2, 3, 4, 5\} = \{4, 2, 3, 1, 5\} = \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 5, 5\} \)

2. \( \{2, 4\} \subseteq \{1, 2, 3, 4, 5\} \) is true

3. \( \{1, 2, 3, 4, 5\} \subseteq \{2, 4\} \) is false
   - E.g., 1 is a counterexample to “if \( x \in \{1, 2, 3, 4, 5\} \), then \( x \in \{2, 4\} \)”

4. \( \emptyset \subseteq \{1, 2, 3, 4, 5\} \) is true
   - There is no counterexample to “if \( x \in \emptyset \), then \( x \in \{1, 2, 3\} \)”
   - The empty set is a subset of every set

5. \( \{\text{John, Sue, Chen, Shankar}\} \) is a set containing 4 names
   - \( U = \) the set of all first names of people

6. \( \{(1, 3), (2, 5), (3, 7)\} \) is a set of ordered pairs

7. \( \{\{3, 4\}, \{5, 6, 7\}\} \) is a set of sets

"bag" contains duplicates

A set considers only unique elements
Set-Builder Notation

Even natural numbers
- \( \{ x : x \in \mathbb{N} \text{ and } x \text{ is even} \} \) or
- \( \{ x \in \mathbb{N} : x \text{ is even} \} \) or
- \( \{ x \in \mathbb{N} : x = 2k \text{ for some } k \in \mathbb{N} \} \) (a property description)
- \( \{ 2k : k \in \mathbb{N} \} \) (a form description)

Intervals
- \( \{ x \in \mathbb{R} : -2.1 \leq x \leq 2.6 \} \) or \([-2.1, 2.6]\)
- \( \{ x \in \mathbb{R} : -2.1 < x < 2.6 \} \) or \((-2.1, 2.6)\)
- \( \{ x \in \mathbb{R} : -2.1 < x \leq 2.6 \} \) or \((-2.1, 2.6]\)
- \( \{ x \in \mathbb{R} : -2.1 \leq x < 2.6 \} \) or \([-2.1, 2.6)\)
- \( \{ x \in \mathbb{N} : 3 \leq x < 6 \} \) or \([3, 6) = \{3, 4, 5\}\)

Other examples: give an alternate description
- \( \{ n \in \mathbb{N} : n \text{ has exactly two positive divisors} \} \)
- \( \{ x \in \mathbb{R} : x^2 + 1 = 0 \} \)
Examples of Form Notation

1. The set of integers that are multiples of 3: \( \{3k : k \in \mathbb{Z}\} \quad \{3, 6, 9, \ldots\} \)

2. The set of perfect square integers: \( \{m^2 : m \in \mathbb{N}\} \) or \( \{m^2 : m \in \mathbb{Z}\} \quad \{0, 1, 4, 9, \ldots\} \)

3. The set of natural numbers that end in a 1: \( \{10k + 1 : k \in \mathbb{N}\} \quad \{1, 11, 21, \ldots\} \)

4. The set \( \mathbb{Q} \): \( \{\frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}^+\} \)
Operations on Sets

Operations on Sets $A$ and $B$

1. Intersection $A \cap B$: $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$
2. Union $A \cup B$: $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$
3. Difference $A - B$: $A - B = \{x \in U : x \in A \text{ and } x \notin B\}$
4. Complement $A'$: $A' = \{x \in U : x \notin A\}$ or $A' = U - A$

Definition

Set $A$ and $B$ are disjoint if $A \cap B = \emptyset$.

Example: Suppose $U = \{1, 2, \ldots, 12\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$

- $A' = \{6, 7, \ldots, 12\}$
- $A \cap B = \{2, 4\}$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$
- $A - B = \{1, 3, 5\}$
- $B - A = \{6, 8, 10\}$
- $A \cap \{8, 10, 12\} = \emptyset$
- $U' = \emptyset$
### BHT for Sets

**Theorem**

For sets $A$, $B$, and $C$, the empty set $\emptyset$, and the universal set $U$, the following properties hold:

<table>
<thead>
<tr>
<th>Property</th>
<th>Condition</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Commutative</td>
<td>$A \cap B = B \cap A$</td>
<td>$A \cup B = B \cup A$</td>
</tr>
<tr>
<td>(b) Associative</td>
<td>$(A \cap B) \cap C = A \cap (B \cap C)$</td>
<td>$(A \cup B) \cup C = A \cup (B \cup C)$</td>
</tr>
<tr>
<td>(c) Distributive</td>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
</tr>
<tr>
<td>(d) Identity</td>
<td>$A \cap U = A$</td>
<td>$A \cup \emptyset = A$</td>
</tr>
<tr>
<td>(e) Negation</td>
<td>$A \cup A' = U$</td>
<td>$A \cap A' = \emptyset$</td>
</tr>
<tr>
<td>(f) Double negative</td>
<td>$(A')' = A$</td>
<td></td>
</tr>
<tr>
<td>(g) Idempotent</td>
<td>$A \cap A = A$</td>
<td>$A \cup A = A$</td>
</tr>
<tr>
<td>(h) DeMorgan’s laws</td>
<td>$(A \cap B)' = A' \cup B'$</td>
<td>$(A \cup B)' = A' \cap B'$</td>
</tr>
<tr>
<td>(i) Universal bound</td>
<td>$A \cup U = U$</td>
<td>$A \cap \emptyset = \emptyset$</td>
</tr>
<tr>
<td>(j) Absorption</td>
<td>$A \cap (A \cup B) = A$</td>
<td>$A \cup (A \cap B) = A$</td>
</tr>
<tr>
<td>(k) Complements</td>
<td>$U' = \emptyset$</td>
<td>$\emptyset' = U$</td>
</tr>
<tr>
<td>(l) Compl. &amp; neg.</td>
<td>$A - B = A \cap B'$</td>
<td></td>
</tr>
</tbody>
</table>

**Duality principle:**

- $U \rightarrow \wedge$  
- $\phi \rightarrow \wedge$  
- $\wedge \rightarrow U$  
- $U \rightarrow \phi$
Verification via Venn Diagrams

Example: \( U = \{1, 2, \ldots, 15, 16\}, A = \{1, 2, 5, 7, 9, 11, 13, 15\}, \)
\( B = \{2, 3, 5, 7, 11, 13\}, C = \{1, 4, 9, 16\} \)

Example: Show that \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)

Also: \( A \cap (B \cup C) \subseteq B \cup C \)
Example 1: Prove that $A \cap B \subseteq A$

1. Let $x \in A \cap B$

2. Then $x \in A$ and $x \in B$

3. In particular, $x \in A$

4. So $(x \in A \cap B) \rightarrow (x \in A)$, and hence $A \cap B \subseteq A$
Example 2: Prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

1. Let $x \in A \cap (B \cup C)$

2. Then $x \in A$ and $x \in B \cup C$

3. Case 1: $x \in B$
   
   3.1 Then $x \in A \cap B$ since $x \in A$ and $x \in B$
   
   3.2 Hence in $(A \cap B) \cup (A \cap C)$ by argument similar to Example 1

4. Case 2: $x \in C$
   
   4.1 Then $x \in A \cap C$
   
   4.2 Hence in $(A \cap B) \cup (A \cap C)$ by argument similar to Example 1

5. In either case, $x \in A \cap (B \cup C)$ implies that $x \in (A \cap B) \cup (A \cap C)$
Example 3: Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- By above result, it suffices to show that 
  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Let $x \in (A \cap B) \cup (A \cap C)$

1. Show that $x \in A$
   - Case 1: $x \in A \cap B$, then $x \in A$ and $x \in B$ hence $x \in A$
   - Case 2: $x \in A \cap C$, then $x \in A$ and $x \in C$, hence $x \in A$
   - So in both cases, $x \in A$

2. Show that $x \in B \cup C$
   - Case 1: $x \in A \cap B$, hence $x \in B$ \( \Rightarrow \) $x \in B \cup C$
   - Case 2: $x \in A \cap C$, hence $x \in C$

3. $x \in A$ and $x \in B \cup C$, hence $x \in A \cap (B \cup C)$
Example 4: Prove that \((A \subseteq B) \land (B \subseteq C) \rightarrow (A \subseteq C)\)

1. Let \(x \in A\)

2. Then \(x \in B\) since \(A \subseteq B\)

3. Hence \(x \in C\) since \(B \subseteq C\)

4. So \((x \in A) \rightarrow (x \in C)\), and hence \(A \subseteq C\)
The Inclusion-Exclusion Principle

Definition

\[ n(A) = \text{the number of elements in the set } A \]

Example

- \( A = \{2k : k \in \mathbb{Z}^+ \text{ and } k \leq 15\} \) and \( B = \{3k : k \in \mathbb{Z}^+ \text{ and } k \leq 10\} \)

- \( n(A) = 7 \quad n(B) = 3 \)

- \( A \cap B = \{2, 4, 6, 10, 12, 14\} \cap \{3, 6, 9\} \)

- \( n(A \cap B) = 1 \text{ since } A \cap B = \{6\} \)

- \( A \cup B = \{2, 3, 4, 6, 9, 10, 12, 14\} \)

- \( n(A \cup B) = 7 + 3 - 1 \)

Theorem (The Inclusion-Exclusion Principle)

Let sets \( A, B, \) and \( C \) be given. Then

\[ n(A \cup B) = n(A) + n(B) - n(A \cap B) \]

\[ n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \]

\[ n(A) + n(B) + n(C) = n_1 + n_2 + n_3 + 2n_4 + 2n_5 + 2n_6 + 3n_7 \]
Cartesian Products

Ordered pairs:

Definition

The Cartesian product $A \times B$ of sets $A$ and $B$ is defined as $\{(a, b) : a \in A \text{ and } b \in B\}$. When both coordinates are taken from the same set $A$, we write $A^2$ instead of $A \times A$.

Example: For second graph, the set of plotted points are $\{(x, y) \in \mathbb{R}^2 : y = 3x - 1\}$

Example: Succinctly describe the sequence $a_1 = 2, a_2 = 4, a_3 = 8, \ldots$ using ordered pairs $(1, 2), (2, 4), (3, 8)$, etc.: $\{(n, 2^n) : n \in \mathbb{Z}^+\}$

Example: Describe all possible (sandwich, drink) orders where sandwiches are of type 1, 2, or 3 and drinks are $A, B, C, \text{ or } D$: $\{1, 2, 3\} \times \{A, B, C, D\}$

Example: Describe all possible pairs (type of A, type of B) of people you might encounter on the island of Liars and Truth-tellers: $(L, T)$, $(T, L)$, $(L, L)$, $(T, T)$

Lecture 11
Size of Cartesian Products

Example: \( A = \{2, 4, 6, 8\} \) and \( B = \{1, 2, 3, 4, 5\} \)

\[
A \times B
\]

<table>
<thead>
<tr>
<th>Elements of ( A )</th>
<th>Elements of ( B ) ( \rightarrow )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 ( \downarrow )</td>
<td>1 (2,1)</td>
</tr>
<tr>
<td>4</td>
<td>2 (2,2)</td>
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<tr>
<td>6</td>
<td>3 (2,3)</td>
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<tr>
<td>8</td>
<td>4 (2,4)</td>
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<tr>
<td></td>
<td>5 (2,5)</td>
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</tbody>
</table>

\[
B \times A
\]

<table>
<thead>
<tr>
<th>Elements of ( B )</th>
<th>Elements of ( A ) ( \rightarrow )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \downarrow )</td>
<td>2 (1,2)</td>
</tr>
<tr>
<td>2</td>
<td>4 (1,4)</td>
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<tr>
<td>3</td>
<td>6 (1,6)</td>
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<tr>
<td>4</td>
<td>8 (1,8)</td>
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<tr>
<td>5</td>
<td>2 (2,2)</td>
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<td>4 (2,4)</td>
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<td></td>
<td>6 (5,6)</td>
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<tr>
<td></td>
<td>8 (5,8)</td>
</tr>
</tbody>
</table>

Theorem

For all finite sets \( A \) and \( B \), \( n(A \times B) = n(A) \cdot n(B) \).
Cartesian Products Gone Wild!

Definition

For any integer \( n \geq 3 \), the structure \((x_1, x_2, \ldots, x_n)\) is called an \( n \)-tuple.

Definition

Given sets \( S_1, S_2, \ldots, S_n \), the Cartesian product \( S_1 \times S_2 \times \cdots \times S_n \) is the set of all \( n \)-tuples \((x_1, x_2, \ldots, x_n)\) such that \( x_1 \in S_1, x_2 \in S_2, \ldots, x_n \in S_n \).

When \( S_1, \ldots, S_n \) are all equal to the same set \( S \), we write

\[
S^n = S \times S \times \cdots \times S.
\]

Theorem A

For any finite sets \( S_1, S_2, \ldots, S_k \), \( n(S_1 \times S_2 \times \cdots \times S_k) = n(S_1) \cdot n(S_2) \cdots \cdot n(S_k) \)
Examples: Cartesian Products and Sets of Sets

Example 1: Truth-table values for \( p, q, r, s \):
\[
S = \{T, F\}^4 = S \times S \times S \times S = 2^4
\]

Example 2: \( 12 \times 12 \) multiplication table, e.g., \((2, 3, 6), (5, 1, 5)\), and so on:
\[
\{ (a, b, c) \in \{S\} \times \{S\} \times \mathbb{N} : a \times b = c \} \cup \{ (a, b, a \times b) : (a, b) \in S^2 \}
\]

Example 3: List 4 elements of
\[
\{ T, F \}^3: (T, T, T), (T, T, F), (T, F, T), (F, F, F)
\]
\[
S = \{ s \in \{0, 1\}^5 : s \text{ consists of three 0's and two 1's in some order} \}
\]
\[
(0, 0, 0, 1, 1), (1, 0, 0, 0, 1)
\]
\[
\{1, 2, 3, 4, 5\} \times \{x, y, z\} \times \{A, B, C, D\}: (1, x, A), (5, z, D)
\]

Example 4: Sets of sets
\[
\text{Is } \{1, 2\} \subseteq \{\{1, 2\}, \{1, 3, 4\}\}? \quad \text{NO! e.g. } 1 \text{ is not an element of the right-hand set}
\]
\[
\text{Is } \{1, 2\} \in \{\{1, 2\}, \{1, 3, 4\}\}? \quad \text{YES}
\]
\[
\text{What is } n(\{\{1, 2\}, \{3, 4\}, \{1, 3, 4\}\})? \quad 3
\]
\[
\text{What element of } S = \{1, 2, 3, \{1, 2\}, \{1, 3, 4\}\} \text{ is also a subset of } S? \quad \{1, 2\}
\]
\[
\text{What are the elements of } S? \quad 1, 2, 3, 5, 1, 2
\]
\[
\text{What is } n(S)? \quad 5
\]
Definition

The power set of $A$ is $\mathcal{P}(A) = \{ S : S \subseteq A \}$.

Example: For $A = \{1, 2, 3, 4\}$, list the elements of $\mathcal{P}(A)$ in an orderly manner.

<table>
<thead>
<tr>
<th>$\mathcal{P}({1})$</th>
<th>$\emptyset$</th>
<th>${1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{P}({1, 2})$</td>
<td>$\emptyset$</td>
<td>${1}$</td>
</tr>
<tr>
<td>$\mathcal{P}({1, 2, 3})$</td>
<td>$\emptyset$</td>
<td>${1}$</td>
</tr>
<tr>
<td>$\mathcal{P}({1, 2, 3, 4})$</td>
<td>$\emptyset$</td>
<td>${1}$</td>
</tr>
</tbody>
</table>

Some questions:

- How large is $\mathcal{P}(A)$?
  - $16$
- If $B = \{2, 4, 6, 8\}$, how large is $\mathcal{P}(B)$?
  - $16$
- If $C = \{1, 2, 3, 4, 5\}$, how large is $\mathcal{P}(C)$?
  - $32$

Theorem B

For any finite set $A$, if $k = n(A)$, then $n(\mathcal{P}(A)) = 2^k$. 