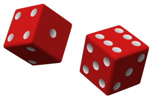
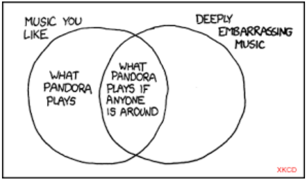
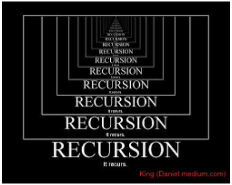


INFO 150: A Mathematical Foundation for Informatics

Peter J. Haas





$$2, 3, 5, 7, 11, 101, 131, 151, ?$$

INFO 150: A Mathematical Foundation for Informatics

Details at the course website: tinyurl.com/INFO150-F19

Teaching Staff

- Instructor: Prof. Peter J. Haas
- TA: Shivam Srivastava
- UCA: Lucy Cousins
- Grader: David Ter-Ovanesyan

What is this introductory course about?

- *Discrete* mathematics and the mathematical method
  - Versus *continuous* mathematics (calculus)
  - Because computers deal with 0's and 1's

Aimed at students in Informatics

- The "outward looking" area of computer science
- Focus on development & application of computational principles and techniques to advance other disciplines
- Prerequisites: A high school math background (algebra! $y^a \times y^b = y^{a+b}$)

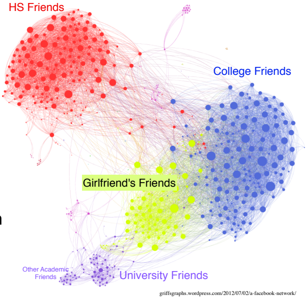
Course Goals and Objectives

Goals

- Learn to think (more) mathematically
- Learn to communicate thinking using mathematical language
- Prep for future courses in computer science

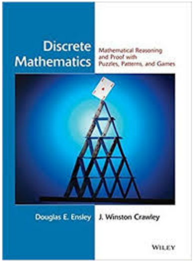
Objectives (see website)

- Recursive thinking: dealing wth complexity
- Mathematical logic: thinking clearly
- Mathematical writing: communicating clearly
- Abstraction: Leveraging what you know
- Functions, relations, sets: basis for programs and data
- Combinatorics: counting things
- Probability: dealing with uncertainty
- Graphs: dealing with relationships



Applications to puzzles, games, programming, and important real-world problems

Course logistics



Textbook: Ensley and Crawley

- Expensive, but can buy used or rent (e.g., eCampus)
- Do not buy *Student Solutions Manual* in place of textbook

Schedules

- Class meetings MW 2:30-3:45 in CompSci 140
  - Attendance is required!
  - In class graded activities (bring paper and pencil!)
  - Lecture topics and readings: see syllabus on website
- My office hours: Mon 4:30-5:30, Wed 10-11, and by appointment
- Shivam's office hours: Tues 10:30-noon, Fri 10:30-noon
- Two evening midterms (7-9pm):
  - Thurs 17 Oct in ILC S331
  - Thurs 14 Nov (location TBD)
- Final exam: Fri 13 Dec, 3:30-5:30 (location TBD)

Lecture slides

- Annotated corrected slides posted after each unit

More Logistics

Piazza for online discussion

- Sign up via webpage link
- Announcements will emailed from Piazza & posted on website
- Ground rules
  - Be respectful, on-topic, and helpful (anonymity allowed—don't abuse!)
  - Hints or clarifications only (don't just ask for or post answers)
  - For private matters, post privately (preferred) or email me.

Academic honesty policy

- See link to UMass policy page on website—ignorance is no excuse! (AIQ quiz)
- Exams: closed book, no outside help (cheating = F)
- In-class assignments: help from classmates & instructor (writeup must be in own words)
- Homework (we will use Gradescope) – sign up now!
  - Can discuss with other students
  - Writeup in your own words: appearance of copying = F
  - External sources (print or web) must be cited
  - No posting of class materials (incl. video/audio recordings) online without prior instructor permission, or providing to third party such as StudySoup

Take notes, then write

Course Requirements and Grading

- Homework/attendance: 40%
- Two midterm exams: 30%
- Final exam: 30%
- Optional Project:
  - Research report on a topic in discrete math
  - 3-5 pages of text exclusive of pictures
  - Report due by end of semester
  - Will push grade up if on boundary
    - More pushing if lower current grade
    - If doing well, won't hurt not to do the project
- Let us know if you are falling behind (Academic Alert...)

Introductions

Me

- Joined UMass in 2017 after 30 years at IBM Research & Stanford University
- Math/CS interests: Data management & analytics, prob/stats, computer simulation
- Real-world applications: air pollution modeling, computational biology (Watson and P53), healthcare
- Random fact: related by marriage to the screenwriter for Star Wars

You?

Mathematics as a Language

A language has two parts:

- Syntax and Grammar: How do we talk about things?
  - Math notation ($a = a_1, a_2, \ldots$; $S = \sum_{i=1}^{k} i^2$; $\mathbf{Y} = \mathbf{X}^\top \mathbf{X}$, etc.)
  - Logic ($\forall x \in \mathbb{N}, \exists y \in \mathbb{N}$ such that $y = x/2$)
- Mathematical objects: What things do we talk about?
  - Numbers (sequences, numerical patterns, series, divisibility, ...)
  - Sets
  - Functions
  - Probabilities
  - Graphs
  - Matrices

We use mathematical language to talk about the real world via abstraction

- $35 =$ approximate number of people in the room
- $S =$ the set of people in the room
- Is a math "sentence" true? (proofs & counter-examples)

Number Puzzles [E&C Section 1.2]

1. 1, 9, 17, 25, 33, 41, ??
2. 1, 4, 9, 16, 25, 36, ??
3. 2, 4, 8, 16, 32, 64, ??
4. 1, 2, 6, 24, 120, 720, ??

Why do we care?

- Training for recursive thinking in a simple setting
- Used later when learning how to write proofs
- Diagnosing time and space complexity of computations
  - "At each time step, each process spawns two more processes"
  - "Each sampling step removes 2/3 of the items and adds 10 more items"
  - "The nth pass through the data has to process n rows of the table

Guess the Next Number

1. 1, 9, 17, 25, 33, 41, ??
2. 1, 4, 9, 16, 25, 36, ??
3. 2, 4, 8, 16, 32, 64, ??
4. 1, 2, 6, 24, 120, 720, ??

Strategy: Look for Patterns

- Relate each term to previous terms (arithmetic formula)
- Describe in terms of position in sequence
- Recognize the set of integers from the examples

Patterns

Example

- Describe the sequence 1, 3, 5, 7, 9, ... each of the three ways

Solution

- Relate each term to previous terms

Each term is 2 more than previous

- Describe in terms of position in sequence

n^{th} term is $2 \cdot n - 1$

- Recognize the set of integers from the examples

The odd natural numbers

Sequences and Sequence Notation

Recursive Formula

Each term is described in relation to previous terms via a recurrence relation

Closed Formula

Each term is described in terms of its position in the sequence

Sequence Notation

Sequence name is a lower-case letter (a, b, ...) and a subscript gives position in sequence: $a_n =$ nth term in sequence a

Example

- $a = 1, 3, 5, 7, 9, \ldots$
- $a_1 = 1,\ a_2 = 3,\ a_5 = 9$
  (it's like a function; subscript = ordinal number)
- Closed formula: $a_n = 2n - 1$ (for $n \geq 1$)
- Recursive formula: $a_1 = 1$ and $a_n = a_{n-1} + 2$ (for $n \geq 2$)

Examples

For the sequence $a_n = 2^n - 1$ with $a_1 = 1$:

- Write the first 3 terms: $a_1 = 1,\ a_2 = 2^2 - 1 = 3,\ a_3 = 2^3 - 1 = 7$
- Value of 10th term: $a_{10} = 2^{10} - 1 = 1024 - 1 = 1023$
- Formula for $(k+1)$st term: $a_{k+1} = 2^{k+1} - 1$
- Formula for $b_i = a_{2i-3}$: $b_i = a_{2i-3} = 2^{2i-3} - 1$ $\left[b_2 = 2^{2\cdot 2-3} - 1 = 2^1 - 1 = 1 \right]$

For the sequence $a_n = a_{n-1} + 5$ with $a_1 = 1$:

- Write the first 3 terms: $a_1 = 1,\ a_2 = a_1 + 5 = 6,\ a_3 = a_2 + 5 = 11$
- Recursive formula for 80th term: $a_{80} = a_{79} + 5$
- Recursive formula for $(k+1)$st term: $a_{k+1} = a_{(k+1)-1} + 5 = a_k + 5$
- Recursive formula for a_{2j-3}: $a_{2j-3} = a_{(2j-3)-1} + 5 = a_{2j-4} + 5$

Discovering Patterns in Sequences

Give Recursive and closed formulas:

recursive form

$+8 \quad +8 \quad +8 \quad \ldots$

1. 1, 9, 17, 25, 33, 41, ?? 49 $\boxed{a_1 = 1 \text{ and } a_n = a_{n-1} + 8}$

b_n: 8, 16, 24, 32

$b_n = 8n$ and $a_n = b_n - 7$, so $\boxed{a_n = 8n - 7,\ n \geq 1}$

2. 1, 4, 9, 16, 25, 36, ?? 49 $\boxed{a_n = n^2,\ n \geq 1}$

$a_n - a_{n-1} = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1,$

so $\boxed{a_1 = 1 \text{ and } a_n = a_{n-1} + 2n - 1}$

(1) look for differences and quotients—how fast do the numbers grow?
(2) compare to simple series with same recurrence

Discovering Patterns in Sequences

Give Recursive and closed formulas:

1. 2, 4, 8, 16, 32, 64, ?? 128 $\boxed{a_n = 2^n,\ n \geq 1}$

$a_n - a_{n-1} = 2^n - 2^{n-1} = 2^{n-1}(2-1) = 2^{n-1}$, so $\boxed{a_1 = 2 \text{ and } a_n = a_{n-1} + 2^{n-1}}$

or note that $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = 2$, so $\frac{a_n}{a_{n-1}} = 2$, so $\boxed{a_1 = 2 \text{ and } a_n = 2a_{n-1}}$

2. 1, 2, 6, 24, 120, 720, ?? 5040

$\frac{a_2}{a_1} = 2$, $\frac{a_3}{a_2} = 3$, $\frac{a_4}{a_3} = 4$, $\frac{a_n}{a_{n-1}} = n$ so $\boxed{a_1 = 1 \text{ and } a_n = n a_{n-1}}$

$a_1 = 1$, $a_2 = 2 \cdot a_1 = 2 \cdot 1$, $a_3 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1$ $\boxed{a_n = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!}$

"n factorial"

(1) look for differences and quotients—how fast do the numbers grow?
(2) compare to simple series with same recurrence

A Rockstar Sequence: Fibonacci Numbers

The Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The recurrence relation

$F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

The closed formula (Binet's formula)

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Applications include (see *Fibonacci Quarterly*):

1. Fibonacci search, Fibonacci heaps
2. Biology and more (leaf/petal patterns, tree branching, ...)

Sums

Notation for sums

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \text{sum of first } n \text{ terms of sequence } a$$

Extended notation for sums

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$ $n-m+1$ terms

Example: Evaluate the sums

- $\sum_{k=1}^{3}(2k-1)$: $1+3+5=9$
- $\sum_{k=3}^{3} k^2$: $3^2=9$
- $\sum_{j=0}^{2} 3^j$: $3^0+3^1+3^2= \quad 1+3+9=13$
- $\sum_{k=1}^{3} \frac{1}{k(k+1)}$: $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$

Sums: More Examples

Notation for sums

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = \text{sum of first } n \text{ terms of sequence } a$$

Examples

- Sum of first 10 numbers in sequence $a_k = 1/k$ with $k \geq 1$ $\quad \sum_{k=1}^{10} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{10}$
- $2 + 4 + 8 + 16 + 32 + 64$ $\quad a_n = 2^n$

$$\text{so} \quad \sum_{n=1}^{6} 2^n$$

Sums: Complexity Example

***k*th pass through database looks at *k* records**

- How many records are looked at by the end of the *n*th pass?

$$S = \sum_{k=1}^{n} k = 1 + 2 + \cdots + n$$

Obtain closed form:

$$S = 1 + 2 + \cdots + (n-1) + n$$
$$S = n + (n-1) + \cdots + 2 + 1$$
$$2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1) = n(n+1)$$

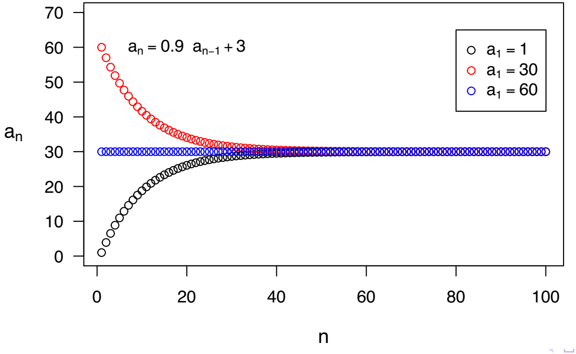
so $\boxed{S = \frac{n(n+1)}{2}}$

Stability of Sequences

Example Try it!

Give the first 4 terms of $a_n = 3a_{n-1} - 6$ with

- $a_1 = 2$:
- $a_1 = 4$:
- $a_1 = 3$:



Q: How to figure out the stable starting value without plotting?