Practice Midterm Solutions

1. Define the following terms
   a) What is a circuit in a graph?
      A sequence of vertices and edges, in the form \(v_1e_1v_2e_2v_3e_3\ldots v_ne_nv_{n+1}\), where the \(v_i\) are vertices in the graph, the \(e_i\) are edges in the graph, and an edge is always listed in between the vertices that serve as its endpoints, and where \(v_1 = v_{n+1}\), and where no edges are repeated.
   b) When are two events independent?
      Events A and B are independent when the probability of A happening does not affect the probability of B happening.
   c) What is combinatorial equivalence?
      Two counting problems in combinatorics have combinatorial equivalence when they have identical answers. Equivalently, there is a one-to-one correspondence between the items being counted in each problem.
   d) What is the rule of sums with overlap?
      The rule states that the number of items in the union of two (possibly overlapping) sets \(A\) and \(B\) equals the number of items in \(A\) plus the number of items in \(B\) minus the number of items in their intersection: \(N(A \cup B) = N(A) + N(B) - N(A \cap B)\). [This rule is closely related to the “general sum rule” in probability, which states that \(P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)\).]
   e) What is a connected graph?
      A graph is connected when there is a walk between any two nodes of the graph.

2. A box contains 40 red balls and 60 blue balls. If we pull out 10 balls at random, what is the probability of getting at least 8 blue balls? What if we draw balls one at a time, and replace each selected ball after recording its color and before the next draw? [You do not need to numerically compute the answers, just write down appropriate formulas.]

**Drawing without replacement:**

This is like forming committees of men and women. There are \(C(100,10)\) possible ways to choose 10 balls regardless of color. To create a set containing exactly \(n\) blue balls, first select \(n\) blue balls and then select \(10 - n\) red balls. This can be done in \(C(60,n)\cdot C(40,10-n)\) ways. So the probability of randomly selecting a set containing exactly \(n\) blue balls is \(C(60,n)\cdot C(40,10-n)/C(100,10)\). Finally, use the sum rule to get

\[
P(8\text{ blue balls}) + P(9\text{ blue balls}) + P(10\text{ blue balls}) = \frac{C(60,8)\cdot C(40,2)}{C(100,10)} + \frac{C(60,9)\cdot C(40,1)}{C(100,10)} + \frac{C(60,10)\cdot C(40,0)}{C(100,10)}
\]

**Drawing with replacement:**

Now the draws are independent, with \(P(\text{blue ball}) = 60/100\) and \(P(\text{red ball}) = 40/100\) in a given draw. The probability of drawing exactly 8 blue balls is the probability of drawing a sequence such as, e.g., RBBBBBBRBB. By the product rule for independent events, the probability of such a sequence is \(\left(\frac{60}{100}\right)^8\left(\frac{40}{100}\right)^2\). There are \(C(10,8)\) such sequences (each having the same probability), since these are the number of ways of choosing the positions (from 1 to 10) of the 8 blue balls (and
then filling the remaining positions with red balls). By the sum rule,

\[ P(8 \text{ blue balls}) = C(10,8) \cdot \left(\frac{60}{100}\right)^8 \left(\frac{40}{100}\right)^2. \]

Applying the sum rule again, we have

\[ P(8 \text{ blue balls}) + P(9 \text{ blue balls}) + P(10 \text{ blue balls}) = C(10,8) \cdot \left(\frac{60}{100}\right)^8 \left(\frac{40}{100}\right)^2 + C(10,9) \cdot \left(\frac{60}{100}\right)^9 \left(\frac{40}{100}\right) + C(10,10) \cdot \left(\frac{60}{100}\right)^{10}. \]

3. A set of 10 cards has 5 black cards and 5 red cards. After shuffling them, what is the probability that the top card is the same color as the bottom card? [Compute the answer numerically; there will be a LOT of cancellation of terms.]

Order is important here, and \( P(\text{the top card being the same color as the bottom card}) = P(\text{both cards being black}) + P(\text{both cards being red}). \)

Let's look first at the probability that both of them are black. There is a total of 10! card arrangements. Of those, how many have a black card as both the first and the last card? That arrangement looks like this: B _ _ _ _ _ _ _ _ B. How many ways are there to arrange that?

Choose the first black card, then choose the last, black card, then arrange the remaining 8 cards in the middle, so \( 5 \cdot 4 \cdot 8! = 20 \cdot 8! \). Therefore, that arrangement can happen with probability \( \frac{20 \cdot 8!}{10!} = \frac{20}{90} = \frac{2}{9} \)

The probability of \( R _ _ _ _ _ _ _ _ R \) is identical, for the total probability is \( \frac{2}{9} + \frac{2}{9} = \frac{4}{9} \).

4. We flip a coin 10 times. What is the probability of the result of the first flip (tails or head) being the same as the result of the last flip?

Note that the coin flips are independent, so what happens on the first and last and flip is completely independent of what happens for the middle flips, and we can focus entirely on the (first flip, last flip) pair. In this case, the two outcomes of interest are (H,H) and (T,T). Because the first and last coin flips are independent, the probability of (H,H) is \( \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \). The probability of (T,T) is also \( \frac{1}{4} \). Thus, using the sum rule, the probability that the first and last tosses produce the same result is \( \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \).

5. Proposition 2 in our textbook’s chapter on graphs states the following: Let \( G \) be a simple, connected graph, and let \( a \) and \( b \) be vertices in \( G \) such that there is no edge between \( a \) and \( b \). If \( G' \) is the graph formed by adding the edge \( (a,b) \) to \( G \), then \( G' \) has a cycle that contains the edge \( (a,b) \). Can this statement be proven by brute force? Explain your answer. Please note: you are not required to prove the statement; you only need to state and explain if the brute force method can be used.

There are an infinite number of graphs that meet the stated conditions. We cannot prove that a proposition about an infinite set is true by brute force, i.e., by looking at every graph. Therefore, this proposition cannot be proven by brute force.

6. Prove the following theorem using induction on the number of edges: In any graph \( G \) having \( n \) edges and \( r \) vertices, the sum of the degrees of the vertices equals \( 2n \), that is, \( \sum_{i=1}^{n} \deg(v_i) = 2n \). [Hint: for the base case of 1 edge, there are only two possible graphs. Next, assume that the theorem holds for all graphs having \( m - 1 \) or fewer edges. For a graph \( G \) having \( m \) edges, remove an edge \( e \) to form a smaller graph \( G' \); and use the induction hypothesis to determine the sum of the degrees of the vertices in \( G' \). Then consider what happens to the sum of the degrees when you add back edge \( e \) to get back the original graph \( G \).]

**Base case:** a graph with one edge. There are only two such graphs, shown below:
In the first of these graphs, A has a degree of one, and B has a degree of one, for a total degree of 2. In the second graph, A has a degree of two (remember that, when computing degrees, loops are counted twice). Therefore, the base case is shown to be true, for graphs with one edge.

**Inductive step:** Let’s say that we have a graph with \( m \) edges. If we take one edge out of the graph, this new graph has \( m - 1 \) edges. By the induction hypothesis we can say that this graph has a sum of degrees equal to \( 2(m - 1) = 2m - 2 \). If we now add back the edge we first took out, this edge either connected two different nodes, or it was a loop on a single node. In the first case the degree of each of these two different nodes has increased by one, for a total sum of degrees of \( 2m - 2 + 1 + 1 = 2m \). In the second case (the edge is a loop), the degree of one node has increased by two, for a total sum of degrees of \( 2m - 2 + 2 = 2m \). Therefore the theorem is proven.

7. Find an Eulerian circuit in the graph below by doing the following:

a) Find a circuit on the graph that, if you take its edges out of the original graph, leaves you with two component graphs. List the nodes of this circuit, in the order they are traversed, and draw the resulting two component graphs.

There are several circuits that we could possibly take out of the graph, but one such circuit is CEFDC. Taking those edges out and deleting the disconnected nodes for the sake of simplicity (as per the example in the textbook) leaves the following graph with two components.
b) Indicate the degree of each of the nodes in the two component graphs.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>2</td>
</tr>
<tr>
<td>I</td>
<td>2</td>
</tr>
<tr>
<td>A</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>K</td>
<td>2</td>
</tr>
<tr>
<td>L</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
</tr>
<tr>
<td>G</td>
<td>2</td>
</tr>
<tr>
<td>H</td>
<td>2</td>
</tr>
</tbody>
</table>

c) Find an Eulerian circuit in each of the component graphs. List the nodes in each of these two Eulerian circuits, in the order they are traversed.

For the left component: CLKBLABJIA. For the right component: FGHF

d) Based on the Eulerian circuit of each of the component graphs, and the circuit in the original graph, list the nodes in the Eulerian circuit of the original graph, in the order they are traversed.

CLKBLABJIAEFGHFDC

8. Prove, by induction on the number of nodes, that a simple connected graph of n nodes has at least \( n - 1 \) edges.

**Base case:** a graph with one node has, at the very least, zero edges. \( 1-1 = 0 \), thus the base case is proven.

**Inductive step:** Assume that a simple connected graph with \( m - 1 \) nodes has at least \( (m-1) - 1 = m - 2 \) edges. If we add a new node, in order for the new graph to be connected, at least one edge has to be added between the new node and a node in the original graph with \( m - 1 \) nodes. Therefore, this new simple connected graph with \( m \) nodes has at least \( (m-2) + 1 = m - 1 \) edges.