SOLUTIONS: MIDTERM EXAMINATION #2

1. (20 points) Given the recursion \( a_n = \frac{9}{2}(a_{n-1} - a_{n-2}) \) with \( a_1 = 6 \) and \( a_2 = 18 \), use induction to prove that \( a_n = 2 \cdot 3^n \) for all positive integers \( n \).

Define \( P(n) \) as the statement “\( a_n = 2 \cdot 3^n \)”, so that \( P(1) \) is “\( a_1 = 2 \cdot 3^1 \)” and \( P(2) \) is “\( a_2 = 2 \cdot 3^2 \)”. Then the proof goes as follows:

1. \( a_1 = 6 = 2 \cdot 3^1 \) and \( a_2 = 18 = 2 \cdot 3^2 \), so that \( P(1) \) and \( P(2) \) are true.
2. Fix \( m \geq 3 \), and assume that \( P(1), P(2), \ldots, P(m-1) \) are all true.
3. Then, using \( P(m-1) \) and \( P(m-2) \),

\[
a_n = \frac{9}{2}(a_{n-1} - a_{n-2}) = \frac{9}{2}(2 \cdot 3^{m-1} - 2 \cdot 3^{m-2}) = \frac{9}{2}(6 \cdot 3^{m-2} - 2 \cdot 3^{m-2}) = \frac{9}{2}(4 \cdot 3^{m-2}) = 2 \cdot 3^m,
\]

so that \( P(m) \) holds.
2. Use induction to prove that \( \sum_{i=1}^{n} (3i - 1) = \frac{n(3n+1)}{2} \) for all positive integers \( n \).

Define \( P(n) \) as the statement “\( \sum_{i=1}^{n} (3i - 1) = \frac{n(3n+1)}{2} \)”, so that \( P(m-1) \) is

“\( \sum_{i=1}^{m-1} (3i - 1) = \frac{(m-1)(3(m-1) + 1)}{2} = \frac{(m-1)(3m-2)}{2} \)”. Then the proof is as follows:

1. For \( n = 1 \), we have \( \sum_{i=1}^{1} (3i - 1) = (3 \cdot 1 - 1) = 2 = \frac{1 \cdot (3 \cdot 1 + 1)}{2} \), so that \( P(1) \) is true.

2. Fix \( m \geq 2 \) and assume that \( P(1), P(2), \ldots, P(m-1) \) are all true.

3. Using \( P(m-1) \), we now have

\[
\sum_{i=1}^{m} (3i - 1) = \left[ \sum_{i=1}^{m-1} (3i - 1) \right] + (3m - 1) = \frac{(m-1)(3m-2)}{2} + (3m - 1) = \frac{(m-1)(3m-2) + 2(m-1)}{2}
\]

\[
= \frac{(3m^2 - 5m + 2) + (6m - 2)}{2} = \frac{3m^2 + m}{2} = \frac{m(3m+1)}{2},
\]

so that \( P(m) \) holds.
3. \( (20\text{ points}) \) Use the Division Theorem to show that if \( n \) is not divisible by 4, then \( n^2 + n \) is even.

Let \( n \) be given. By the Division Theorem, there are three possible cases:

**Case 1:** \( n = 4k + 1 \) for some integer \( k \). Then
\[
 n^2 + n = (4k + 1)^2 + (4k + 1) = (16k^2 + 8k + 1) + (4k + 1) = 16k^2 + 12k + 2 = 2 \cdot (8k^2 + 6k + 1)
\]
and hence is divisible by 2.

**Case 2:** \( n = 4k + 2 \) for some integer \( k \). Then
\[
 n^2 + n = (4k + 2)^2 + (4k + 2) = (16k^2 + 16k + 4) + (4k + 2) = 16k^2 + 20k + 6 = 2 \cdot (8k^2 + 10k + 3)
\]
and hence is divisible by 2.

**Case 3:** \( n = 4k + 3 \) for some integer \( k \). Then
\[
 n^2 + n = (4k + 3)^2 + (4k + 3) = (16k^2 + 24k + 9) + (4k + 3) = 16k^2 + 28k + 12 = 2 \cdot (8k^2 + 14k + 6)
\]
and hence is divisible by 2.

So \( n^2 + n \) is divisible by 2 in all cases.
4. Consider the identity \((A' \cap B) \cup (A \cap B) = B\)

a) \((5\text{ points}) \) Verify the identity using Venn diagrams.

\[ \vspace{2em} A' \cap B \quad \vspace{2em} A \cap B \quad \vspace{2em} (A' \cap B) \cup (A \cap B) \]

b) \((15\text{ points}) \) Prove the identity using an element-wise proof.

(i) We first show that \((A' \cap B) \cup (A \cap B) \subseteq B\).

Let \(x \in (A' \cap B) \cup (A \cap B)\) be given. There are two cases to consider:

Case 1: \(x \in (A' \cap B)\). This implies in particular that \(x \in B\).

Case 2: \(x \in (A \cap B)\). This implies in particular that \(x \in B\).

Thus in either case, we have \(x \in B\), which proves (i).

(ii) We next show that \(B \subseteq (A' \cap B) \cup (A \cap B)\).

Let \(x \in B\) be given. There are two cases.

Case 1: \(x \in A'\), which implies that \(x \in A' \cap B\).

Case 2: \(x \in A\), which implies that \(x \in A \cap B\).

So in all cases, either \(x \in A' \cap B\) or \(x \in A \cap B\), which implies that \(x \in (A' \cap B) \cup (A \cap B)\). This proves (ii).

Finally, (i) and (ii) together prove the desired result.
5. More functions and sets: let $A = \{ x \in \mathbb{R} : x > -1 \}$ and $B = \{ x \in \mathbb{R} : 0 < x \leq 1 \}$, and answer the following questions.

   a) (5 points) Compute the set $\mathcal{P}(\{1,2\}) \times \{ u, v \}$, where $\mathcal{P}(S)$ denotes the power set of a set $S$.

      We have $\mathcal{P}(\{1,2\}) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$, so that $\mathcal{P}(\{1,2\}) \times \{ u, v \} = \{ (\emptyset, u), (\{1\}, u), (\{2\}, u), (\{1,2\}, u), (\{1\}, v), (\{2\}, v), (\{1,2\}, v) \}$.

   b) (5 points) For the function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ with rule $f(x) = x^2$ and the function $g : A \to \mathbb{R}^{\geq 0}$ with rule $g(x) = \frac{1}{1+x}$, give the domain, codomain, and rule for the composite function $h = g \circ f$.

      The domain is $\mathbb{R}$ and the codomain is $\mathbb{R}^{\geq 0}$. The rule is given by $h(x) = g(f(x)) = g(x^2) = \frac{1}{1+x^2}$.

   c) (10 points) For the function $h$ defined in part (b), either give the domain, codomain, and rule for the inverse function $h^{-1}$ or explain why $h$ is not invertible. Does your answer change if the domain and codomain of $g$ are modified to be $\mathbb{R}^{\geq 0}$ and $B$, respectively?

      The function $h$ is not invertible. If we try to invert $h$, we see that the domain of $h^{-1}$ would have to be $\mathbb{R}^{\geq 0}$ and the codomain would have to be $\mathbb{R}$ (since we swap the domain and codomain). Solving the equation $y = \frac{1}{1+x^2}$, we see that the rule would have to be $h^{-1}(y) = \sqrt{\frac{1-y}{y}}$, which is not defined when $y > 1$. If we redefine the domain and codomain of $g$ as suggested, then the domain of $h^{-1}$ is $B$ and the codomain is $\mathbb{R}$. The rule is still $h^{-1}(y) = \sqrt{\frac{1-y}{y}}$, but now $h^{-1}(y)$ is defined for all $y$ in the domain $B$. 