SOLUTIONS: MIDTERM EXAMINATION #1

1. **(15 points)** Define the following terms
   
   a. Predicate

       A statement containing one or more variables that is unambiguously true or false when a value is assigned to each variable.

   b. $a$ is divisible by $b$

       $a$ can be written in the form $a = q \cdot b$ for some integer $q$

   c. Set

       A set is a collection of objects, called members or elements.

   d. The two quantifiers

       The symbols $\forall$ and $\exists$, or “for all” and “there exists”

   e. Counterexample

       An example illustrating that a “for all” statement is false.
2. **(25 points)** Translate these statements. If a statement is in English translate it to mathematical notation, if it is mathematical notation translate it to English. A set $B$ of bus routes contains routes $a$, $b$, and $c$, among others. If $x$ is a bus route, the predicate $L(x)$ means “route $x$ stops at the library”, $G(x)$ means “route $x$ stops at the gym.” There is also set $T$ of bus types. If $x$ is a bus route and $y$ is a bus type, the predicate $U(x,y)$ means “route $x$ uses bus type $y$”, and the predicate $S(x,y)$ means “routes $x$ and $y$ use the same bus type.”

a. Route $a$ stops at the gym but does not stop at the library.

$$G(a) \land \neg L(a)$$

b. If routes $a$ and $b$ both stop at the gym, then they don’t use the same bus type.

$$(G(a) \land G(b)) \rightarrow \neg S(a,b)$$

c. All of the bus routes that stop at the library are of a single bus type.

$$\exists y \in T, \forall x \in B, L(x) \rightarrow U(x,y) \text{ or will accept } \forall x, y \in B, L(x) \land L(y) \rightarrow S(x,y)$$

d. $\neg \exists x \in B, G(x) \land \neg L(x)$

There does not exist a route that stops just at the gym and not at the library.

e. $\forall y \in T, \exists x \in B, U(x,y)$

Every bus type is used by at least one bus route.
3. (15 points) Use truth tables to check if the following statements are logically equivalent:

\((-p \land q) \lor (p \land \neg q)\) and \((-p \land \neg q)\)

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So statements are not logically equivalent

4. (20 points) Solve the following sequence problems.

a. For \(a_n = 4n - 2\), calculate \(a_{k-1}\) and \(a_{k+1}\) and simplify the expression

\[
a_{k-1} = 4(k - 1) - 2 = 4k - 6
\]
\[
a_{k+1} = 4(k + 1) - 2 = 4k + 2
\]

b. Given the recursive formula \(a_n = 2a_{n-1} + 3n\), write a recursive expression for \(a_{3k+1}\), simplifying when possible.

\[
a_{3k+1} = 2a_{(3k+1)-1} + 3(3k + 1) = 2a_{3k} + 9k + 3
\]

c. Give an algebraic expression for the \(n\)th term in the progression:

\(a, \ a + 4, \ a + 8, \ a + 12\)

\(a + 4(n - 1)\)
d. For the sequence in part (b), if $a_5 = 139$, what is $a_6$?

$$a_6 = 2a_5 + 3 \cdot 6 = 2 \cdot 139 + 18 = 296$$

5. **(25 points)** Answer the following miscellaneous questions.

a. Write the negation of the following statement as an English sentence:

“For every integer $x$, there is an integer $y$ that is bigger than $x$.”

There exists an integer $x$ that is greater than or equal to all other integers.

b. Write the negation of the following formula symbolically, so that the resulting formula contains at least one instance of the $\lor$ operator.

$$\forall x, \exists y, R(x, y) \rightarrow P(x) \land Q(y)$$

$$\exists x, \forall y, R(x, y) \land (\neg P(x) \lor \neg Q(x))$$

c. Write (in English) is the inverse of the statement:

“If you don’t chase sticks, then you are not a dog”

If you chase sticks, then you are a dog.

d. Prove the following: For any odd integer $n$, the quantity $n(n+2)+5$ is even.

1. Let an odd integer $n$ be given
2. Then $n$ can be written as $n = 2L + 1$ for some integer $L$
3. Then $n(n+2)+5 = (2L + 1)(2L + 3) + 5 = 4L^2 + 8L + 8 = 2(2L^2 + 4L + 4)$
4. Since $2L^2 + 4L + 4$ is an integer (by closure), $n(n+2)+5$ can be written as $2 \cdot (an integer)$, and hence $n(n+2)+5$ is even