Discrete-Event Systems and Generalized Semi-Markov Processes

Reading: Section 1.4 in Shedler or Section 4.1 in Haas

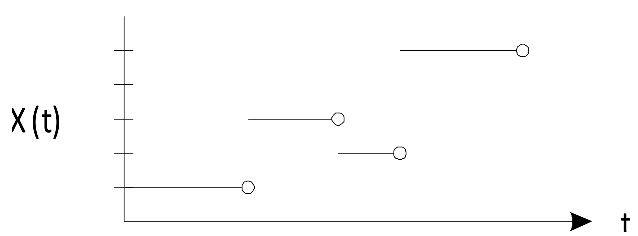
Peter J. Haas

CS 590M: Simulation
Spring Semester 2020

Discrete-Event Systems and Generalized Semi-Markov Processes
- Discrete-Event Stochastic Systems
- The GSMP Model
- Simulating GSMPs
- Generating Clock Readings: Inversion Method
- Markovian and Semi-Markovian GSMPs

Discrete-Event Stochastic Systems

Stochastic state transitions occur at an increasing sequence of random times



How to model underlying process $(X(t) : t \geq 0)$?

- Generalized semi-Markov processes (GSMPs)
- Basic model of a discrete-event system

GSMP Overview

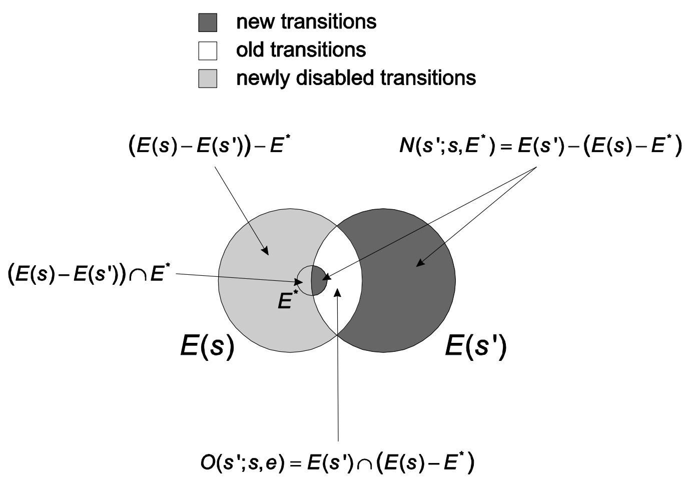
- Events associated with a state "compete" to trigger next state transition
- Each event has own distribution for determining the next state
- New events
  - Associated with new state but not old state, *or*
  - Associated with new state and just triggered state transition
  - Clock is set with time until event occurs (runs down to 0)
- Old events
  - Associated with old and new states, did not trigger transition
  - Clock continues to run down
- Canceled events
  - Associated with old state, but not new state
  - Clock reading is discarded
- Clocks can run down at state-dependent speeds

Clock-Reading Plot

GSMP Building Blocks

- S: a (finite or countably infinite) set of states
- $E = \{e_1, e_2, \ldots, e_M\}$: a finite set of events
- $E(s) \subseteq E$: the set of active events in state $s \in S$
- $p(s'; s, E^*)$: probability that new state $= s'$ when events in E^* simultaneously occur in s
  - Write $p(s'; s, e^*)$ if $E^* = \{e^*\}$ (unique trigger event)
- $r(s, e)$: the nonnegative finite speed at which clock for e runs down in state s
  - Typically $r(s, e) = 1$
  - Set $r(s, e) = 0$ to model "preempt resume" service discipline
- $F(\cdot\,; s', e', s, E^*)$: cdf of new clock-reading for e' after state transition $s \xrightarrow{E^*} s'$
- μ: initial distribution for state and clock readings
  - Assume initial state $s \overset{\mathrm{D}}{\sim} \nu$ and clock readings $\overset{\mathrm{D}}{\sim} F_0(\cdot\,; e, s)$

New and Old Events



Example: GI/G/1 Queue

- **Assume that interarrival-time dist'n F_a and service-time dist'n F_s are continuous (no simult. event occurrences)**
- **Assume that at time $t = 0$ a job arrives to an empty system**

$X(t) = \#$ **of jobs in service or waiting in queue at time** t

Can define $(X(t) : t \geq 0)$ as a GSMP:

- $S =$
- $E =$
- $E(s) =$
- p:
- $F(x; s', e', s, e^*)$:
- $r(s, e) =$
- Initial dist'n:

A More Complex Example: Patrolling Repairman

See handout for details

- Provides an example of how to concisely express GSMP building blocks

Specifying a GSMP can be complex and time-consuming, so why do it?

- Direct guidance for coding (helps catch "corner cases")
- Communicates model at high level (vs poring through code)
- Theory for GSMPs can help in establishing important properties of the simulation
  - Stability (i.e., convergence to steady state), so that steady-state estimation problems are well defined
  - Validity of specific simulation output-analysis methods, so that estimates are correct

GSMPs and GSSMCs

GSMP formally defined in terms of GSSMC $((S_n, C_n) : n \geq 0)$

- S_n = state just after nth transition
- $C_n = (C_{n,1}, C_{n,2}, \ldots, C_{n,M})$ = clock readings just after nth transition
- See Haas or Shedler books for definition of $P\big((s,c), A\big)$ and μ

GSMP Definition

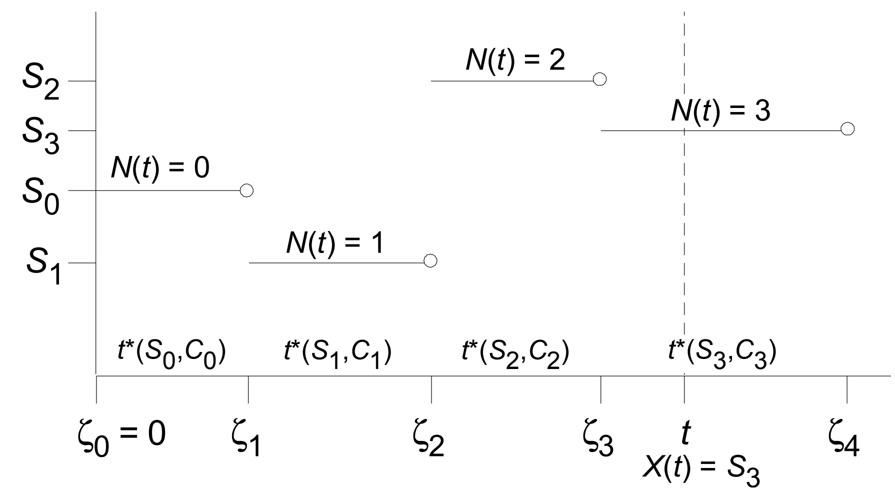
Define

- Holding time: $t^*(s,c) = \min_{\{i: e_i \in E(s)\}} c_i / r(s, e_i)$
- nth state-transition time: $\zeta_n = \sum_{k=0}^{n-1} t^*(s,c)$
- $\#$ of state transitions in $[0,t]$: $N(t) = \max\{n \geq 0 : \zeta_n \leq t\}$

Let $\Delta \notin S$ and set

$$X(t) = \begin{cases} S_{N(t)} & \text{if } N(t) < \infty; \\ \Delta & \text{if } N(t) = \infty \end{cases}$$

GSMP Definition in a Picture



Discrete-Event Systems and Generalized Semi-Markov Processes

Discrete-Event Stochastic Systems

The GSMP Model

Simulating GSMPs

Generating Clock Readings: Inversion Method

Markovian and Semi-Markovian GSMPs

Sample Path Generation

GSMP Simulation Algorithm (Variable Time-Advance)

1. (Initialization) Select $s \stackrel{D}{\sim} \nu$. For each $e_i \in E(s)$ generate a clock reading $c_i \stackrel{D}{\sim} F_0(\,\cdot\,; e_i, s)$. Set $c_i = 0$ for $e_i \notin E(s)$.
2. Determine holding time $t^*(s,c)$ and set of trigger events $E^* = E^*(s,c) = \{e_i : c_i/r(s,e_i) = t^*(s,c)\}$.
3. Generate next state $s' \stackrel{D}{\sim} p(\,\cdot\,; s, E^*)$.
4. For each $e_i \in N(s'; s, E^*)$, generate $c_i' \stackrel{D}{\sim} F(\,\cdot\,; s', e_i, s, E^*)$.
5. For each $e_i \in O(s'; s, E^*)$, set $c_i' = c_i - t^*(s,c)\,r(s,e_i)$.
6. For each $e_i \in (E(s) - E^*) - E(s')$, set $c_i' = 0$ (i.e., cancel event e_i).
7. Set $s = s'$ and $c = c'$, and go to Step 2. (Here $c = (c_1, c_2, \ldots, c_M)$ and similarly for c'.)

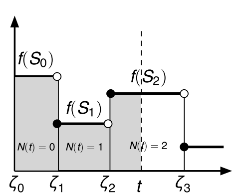
Sample Path Generation, Continued

Algorithm generates sequence of states $(S_n : n \ge 0)$, **clock-reading vectors** $(C_n : n \ge 0)$, **and holding times** $\big(t^*(S_n, C_n) : n \ge 0\big)$

Transition times $(\zeta_n : n \ge 0)$ **and continuous-time process** $\big(X(n) : n \ge 0\big)$ **computed as described previously**

Use usual techniques to estimate quantities like $E\left[f\big(X(t)\big)\right]$ **or even**

$$\alpha = E\left[\frac{1}{t}\int_0^t f\big(X(u)\big)\,du\right]$$
$$= E\left[\frac{1}{t}\left(\sum_{n=0}^{N(t)-1} f(S_n)t^*(S_n, C_n) + f(S_{N(t)})\,(t - \zeta_{N(t)})\right)\right]$$



Flow charts and diagrams can be helpful (see Law, p. 30–32 for an example)

Discrete-Event Systems and Generalized Semi-Markov Processes

Discrete-Event Stochastic Systems

The GSMP Model

Simulating GSMPs

Generating Clock Readings: Inversion Method

Markovian and Semi-Markovian GSMPs

Generating Clock Readings: Example

Exponential distribution with rate (intensity) λ

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases}$$

Mean $= 1/\lambda$

Claim:

If $U \overset{D}{\sim}$ Uniform$(0,1)$ and $V = \frac{-\ln U}{\lambda}$, then $V \overset{D}{\sim} \exp(\lambda)$

Proof:

The Inversion Method: Special Case

Spose that cdf $F(x) = P(V \leq x)$ is increasing and continuous

Claim:

If $U \overset{D}{\sim}$ Uniform$(0,1)$ and $V = F^{-1}(U)$, then $V \overset{D}{\sim} F$

Proof:

Example: Exponential Distribution

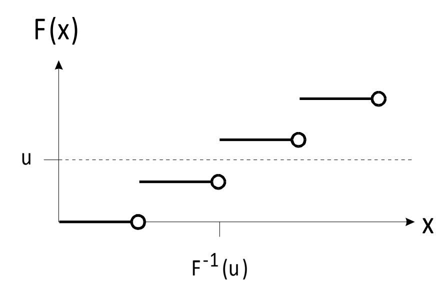
$F(x) = 1 - e^{-\lambda x}$

$F^{-1}(u) =$

The Inversion Method: General Case

Generalized inverse

$F^{-1}(u) = \min\{x : F(x) \geq u\}$



Claim still holds: $F^{-1}(u) \leq x \;\Leftrightarrow\; u \leq F(x)$ **by definition**

Exercise: Show that inversion method $=$ naive method for discrete RVs

Discrete-Event Systems and Generalized Semi-Markov Processes

Discrete-Event Stochastic Systems

The GSMP Model

Simulating GSMPs

Generating Clock Readings: Inversion Method

Markovian and Semi-Markovian GSMPs

Markovian GSMPs

Properties of the Exponential Distribution

If $X \overset{D}{\sim} \exp(\lambda)$ and $Y \overset{D}{\sim} \exp(\mu)$ then

1. $\min(X, Y) \overset{D}{\sim} \exp(\lambda + \mu)$ [indep. of whether min $= X$ or Y]
2. $P(X < Y) = \frac{\lambda}{\lambda+\mu}$
3. $P(X > a + b \mid X > a) = e^{-\lambda b}$ [memoryless property]

Properties 1 and 2 generalize to multiple exponential RVs

Simple GSMP event e'

$F(\cdot\,; s', e', s, E^*) \equiv F(\cdot\,; e')$ and $F_0(\cdot\,; e'; s) \equiv F(\cdot\,; e')$

Markovian GSMPs, Continued

Suppose that all events in a GSMP are simple with exponential clock-setting distn's

Key observation: By memoryless property, whenever GSMP jumps into a state s, clock readings for events in $E(s)$ are mutually independent and exponentially distributed

Simplified Simulation Algorithm (No clock readings needed)

1. (Initialization) Select $s \overset{D}{\sim} \nu$
2. Generate holding time $t^* \overset{D}{\sim} \exp(\lambda)$, where $\lambda = \lambda(s) = \sum_{e_i \in E(s)} \lambda_i$
3. Select $e_i \in E(s)$ as trigger event with probability λ_i/λ
4. Generate the next state $s' \overset{D}{\sim} p(\,\cdot\,; s, e_i)$
5. Set $s = s'$ and go to Step 2

Markovian GSMPs, Continued

Structure of a Markovian GSMP

- Sequence $(S_n : n \geq 0)$ is a DTMC with transition matrix $R(s, s') = \sum_{e_i \in E(s)} p(s'; s, e_i)(\lambda_i/\lambda)$
- Given $(S_n : n \geq 0)$, holding times are mutually independent with holding time in $S_n \overset{D}{\sim} \exp(\lambda(S_n))$

Often, occurrence of e_i in s causes state to change to a unique state $y_i = y_i(s)$ with probability 1

Super-Simplified Simulation Algorithm

1. (Initialization) Select $s \overset{D}{\sim} \nu$
2. Generate holding time $t^* \overset{D}{\sim} \exp(\lambda)$, where $\lambda = \sum_{e_i \in E(s)} \lambda_i$
3. Set $s' = y_i(s)$ with probability λ_i/λ
4. Set $s = s'$ and go to Step 2

Markovian GSMPs, Continued

A GSMP $(X(t) : t \geq 0))$ with simple, exponential transitions is a continuous-time Markov chain (CTMC) [Ross, Ch. 6]

- Finite or countable state space
- Continuous-time Markov property

$$P\big(X(t+u) = s \mid X(s) : 0 \leq s \leq t\big) = P\big(X(t+u) = s \mid X(t)\big)$$

All CTMCs have foregoing structure

- State sequence is a DTMC
- Holding times mutually independent and $\exp\big(\lambda(s)\big)$ in state s

Q: What can go wrong if events are not simple?

Example of Markovian GSMP: Poisson Process

Definition of Poisson process $\big(N(t) : t \geq 0\big)$ with rate λ

- $S = \{0, 1, 2, \ldots\}$
- Single $\exp(\lambda)$ event
- $p(s+1; s, e) = 1$

Can show that

$$P\big(N(t+s) = m+n \mid N(t) = m\big) = \frac{e^{-\lambda s}(\lambda s)^n}{n!}$$

Examples: # arrivals to a queue, # of lightbulb replacements

Semi-Markovian GSMPs

GSMP $\big(X(t) : t \geq 0\big)$ with simple events such that $|E(s)| = 1$ for all $s \in S$ is a semi-Markov process

Definition of semi-Markov process

- Discrete state space S
- State sequence $(X_n : n \geq 0)$ is a DTMC with transition matrix, say, R
- Holding time in $s \overset{D}{\sim} F(\cdot\,; s)$
- "Markov property holds only at state-transition times"

Example: Renewal counting process

- $S = \{0, 1, 2, \ldots\}$
- $R(s, s+1) = 1$ for all $s \in S$
- $F(\cdot\,; s) \equiv G(\cdot)$ for some G