Discrete-Event Systems and Generalized Semi-Markov Processes

Reading: Section 1.4 in Shedler or Section 4.1 in Haas

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CS 590M: Simulation
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Discrete-Event Systems and Generalized Semi-Markov Processes

Discrete-Event Stochastic Systems

The GSMP Model

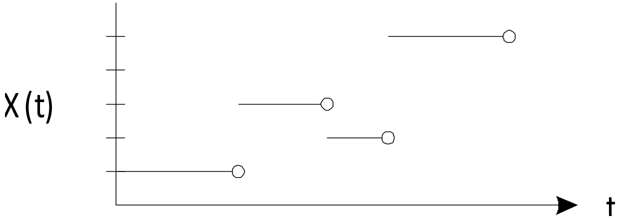
Simulating GSMPs

Generating Clock Readings: Inversion Method

Markovian and Semi-Markovian GSMPs

Discrete-Event Stochastic Systems

Stochastic state transitions occur at an increasing sequence of random times



How to model underlying process $\big(X(t) : t \geq 0\big)$**?**

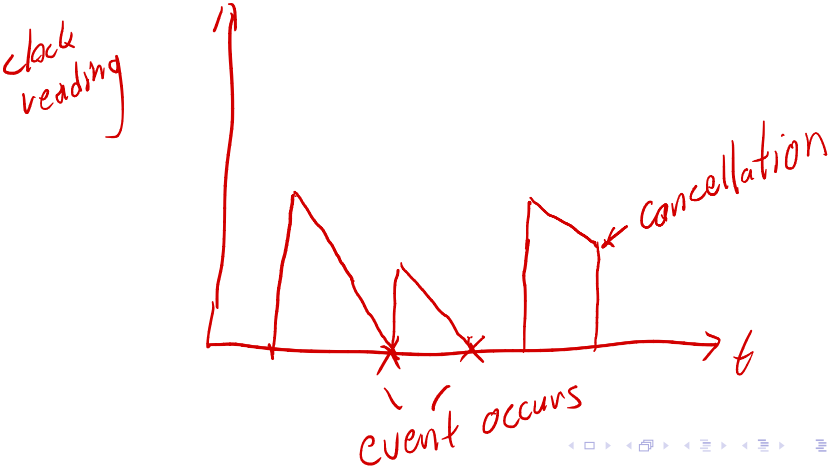
- Generalized semi-Markov processes (GSMPs)
- Basic model of a discrete-event system

GSMP Overview

- Events associated with a state "compete" to trigger next state transition
- Each event has own distribution for determining the next state
- New events
  - Associated with new state but not old state, *or*
  - Associated with new state and just triggered state transition
  - Clock is set with time until event occurs (runs down to 0)
- Old events
  - Associated with old and new states, did not trigger transition
  - Clock continues to run down
- Canceled events
  - Associated with old state, but not new state
  - Clock reading is discarded
- Clocks can run down at state-dependent speeds

Clock-Reading Plot

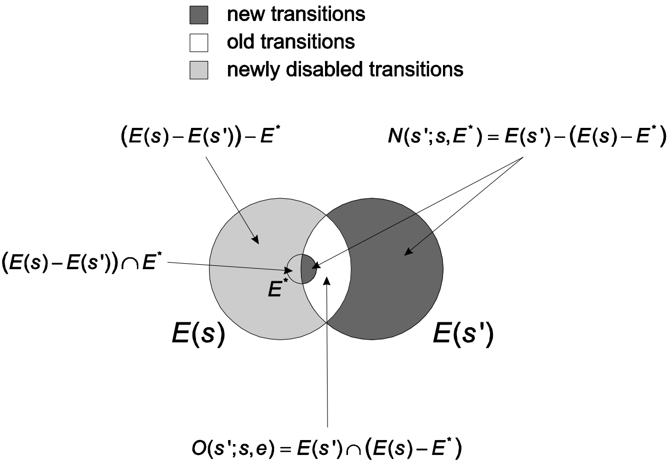
For a given clock:



GSMP Building Blocks

- S: a (finite or countably infinite) set of states
- $E = \{e_1, e_2, \ldots, e_M\}$: a finite set of events
- $E(s) \subseteq E$: the set of active events in state $s \in S$
- $p(s'; s, E^*)$: probability that new state $= s'$ when events in E^* simultaneously occur in s
  - Write $p(s'; s, e^*)$ if $E^* = \{e^*\}$ (unique trigger event)
- $r(s, e)$: the nonnegative finite speed at which clock for e runs down in state s
  - Typically $r(s, e) = 1$
  - Set $r(s, e) = 0$ to model "preempt resume" service discipline
- $F(\,\cdot\,; s', e', s, E^*)$: cdf of new clock-reading for e' after state transition $s \xrightarrow{E^*} s'$
- μ: initial distribution for state and clock readings
  - Assume initial state $s \overset{D}{\sim} \nu$ and clock readings $\overset{D}{\sim} F_0(\,\cdot\,; e, s)$

New and Old Events



Example: GI/G/1 Queue

- **Assume that interarrival-time dist'n F_a and service-time dist'n F_s are continuous (no simult. event occurrences)**
- **Assume that at time $t = 0$ a job arrives to an empty system**

$X(t) = \#$ **of jobs in service or waiting in queue at time** t

Can define $(X(t) : t \geq 0)$ as a GSMP:

- $S = \{0, 1, 2, \ldots\}$
- $E = \{e_1, e_2\}$ $e_1 =$ "arrival", $e_2 =$ "completion of service"
- $E(s) = \{e_1\}$ if $s = 0$; $E(s) = \{e_1, e_2\}$ if $s > 0$
- p: $p(s+1; s, e_1) = 1$, $p(s-1; s, e_2) = 1$, $p(s'; s, e) = 0$ otherwise
- $F(x; s', e', s, e^*)$: $F_a(x)$ if $e' = e_1$ and $F_s(x)$ if $e' = e_2$
- $r(s, e) = 1$ for all s, e
- Initial dist'n: $\nu(1) = 1$, $\nu(s) = 0$ $\forall s \neq 1$; $F_0(x; e_1, s) = F_a(x)$, $F_0(x; e_2, s) = F_s(x)$

A More Complex Example: Patrolling Repairman

See handout for details

- Provides an example of how to concisely express GSMP building blocks

Specifying a GSMP can be complex and time-consuming, so why do it?

- Direct guidance for coding (helps catch "corner cases")
- Communicates model at high level (vs poring through code)
- Theory for GSMPs can help in establishing important properties of the simulation
  - Stability (i.e., convergence to steady state), so that steady-state estimation problems are well defined
  - Validity of specific simulation output-analysis methods, so that estimates are correct

GSMPs and GSSMCs

events $E = \{e_1, \ldots, e_M\}$

GSMP formally defined in terms of GSSMC $\big((S_n, C_n) : n \geq 0\big)$

- $S_n =$ state just after nth transition
- $C_n = (C_{n,1}, C_{n,2}, \ldots, C_{n,M}) =$ clock readings just after nth transition
- See Haas or Shedler books for definition of $P\big((s, c), A\big)$ and μ

GSMP Definition

clock reading for e_i

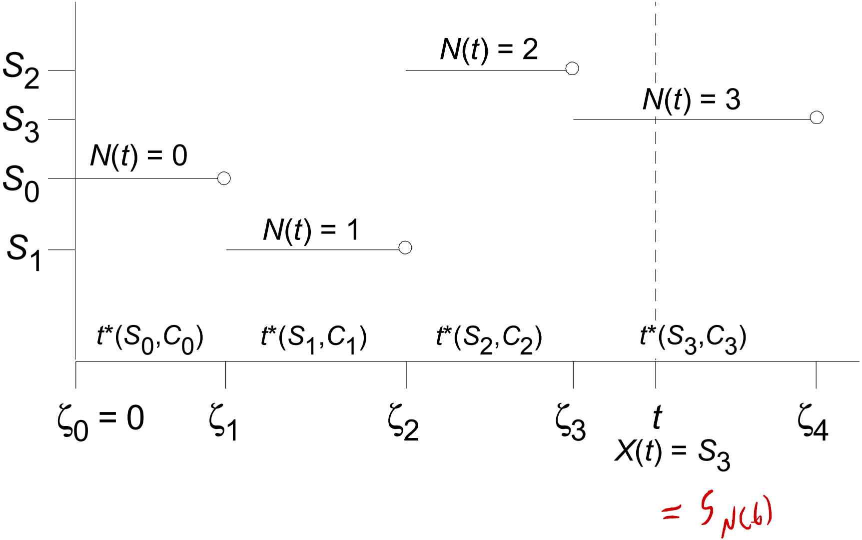
Define

- Holding time: $t^*(s,c) = \min_{\{i : e_i \in E(s)\}} c_i / r(s, e_i)$
- nth state-transition time: $\zeta_n = \sum_{k=0}^{n-1} t^*(s,c)$
- $\#$ of state transitions in $[0,t]$: $N(t) = \max\{n \geq 0 : \zeta_n \leq t\}$

Let $\Delta \notin S$ and set

$$X(t) = \begin{cases} S_{N(t)} & \text{if } N(t) < \infty; \\ \Delta & \text{if } N(t) = \infty \end{cases}$$

GSMP Definition in a Picture



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Sample Path Generation

GSMP Simulation Algorithm (Variable Time-Advance)

1. (Initialization) Select $s \stackrel{D}{\sim} \nu$. For each $e_i \in E(s)$ generate a clock reading $c_i \stackrel{D}{\sim} F_0(\,\cdot\,; e_i, s)$. Set $c_i = 0$ for $e_i \notin E(s)$.
2. Determine holding time $t^*(s, c)$ and set of trigger events $E^* = E^*(s, c) = \{e_i : c_i/r(s, e_i) = t^*(s, c)\}$.
3. Generate next state $s' \stackrel{D}{\sim} p(\,\cdot\,; s, E^*)$.
4. For each $e_i \in N(s'; s, E^*)$, generate $c_i' \stackrel{D}{\sim} F(\,\cdot\,; s', e_i, s, E^*)$.
5. For each $e_i \in O(s'; s, E^*)$, set $c_i' = c_i - t^*(s, c)\, r(s, e_i)$.
6. For each $e_i \in (E(s) - E^*) - E(s')$, set $c_i' = 0$ (i.e., cancel event e_i).
7. Set $s = s'$ and $c = c'$, and go to Step 2. (Here $c = (c_1, c_2, \ldots, c_M)$ and similarly for c'.)

Sample Path Generation, Continued

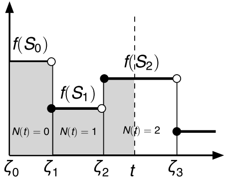
Algorithm generates sequence of states $(S_n : n \ge 0)$, clock-reading vectors $(C_n : n \ge 0)$, and holding times $\left(t^*(S_n, C_n) : n \ge 0\right)$

Transition times $(\zeta_n : n \ge 0)$ and continuous-time process $\left(X(t) : t \ge 0\right)$ computed as described previously

Use usual techniques to estimate quantities like $E\left[f\left(X(t)\right)\right]$ or even

$$\alpha = E\left[\frac{1}{t}\int_0^t f\left(X(u)\right)du\right]$$

$$= E\left[\frac{1}{t}\left(\sum_{n=0}^{N(t)-1} f(S_n)t^*(S_n, C_n) + f\left(S_{N(t)}\right)\left(t - \zeta_{N(t)}\right)\right)\right]$$



Flow charts and diagrams can be helpful (see Law, p. 30–32 for an example)

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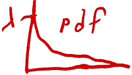
Generating Clock Readings: Example



Exponential distribution with rate (intensity) λ

pdf $f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases}$ and cdf $F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0 \end{cases}$

Mean $= 1/\lambda$

Claim:

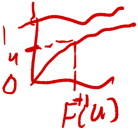
If $U \overset{D}{\sim} \text{Uniform}(0,1)$ and $V = \frac{-\ln U}{\lambda}$, then $V \overset{D}{\sim} \exp(\lambda)$

Proof:

$$P(V > x) = P\left(-\frac{\ln U}{\lambda} > x\right) = P(\ln U < -\lambda x)$$

$$= P(U < e^{-\lambda x}) = e^{-\lambda x}$$

The Inversion Method: Special Case



Spose that cdf $F(x) = P(V \leq x)$ is increasing and continuous

> Claim:
>
> If $U \overset{D}{\sim} \text{Uniform}(0,1)$ and $V = F^{-1}(U)$, then $V \overset{D}{\sim} F$

Proof:

$$P(V \leq x) = P(F^{-1}(U) \leq x) = P\big(F(F^{-1}(U)) \leq F(x)\big)$$

$$= P(U \leq F(x)) = F(x)$$

Example: Exponential Distribution

$$F(x) = 1 - e^{-\lambda x}$$

$$F^{-1}(u) = -\frac{\ln(1-u)}{\lambda}$$

$$= -\frac{\ln(u')}{\lambda}$$

$u' = 1 - u$

$1 - e^{-\lambda x} = u$

$e^{-\lambda x} = 1 - u$

$-\lambda x = \ln(1-u)$

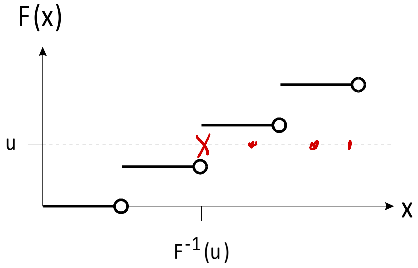
$x = -\frac{\ln(1-u)}{\lambda}$

special case of inversion method

The Inversion Method: General Case

Generalized inverse

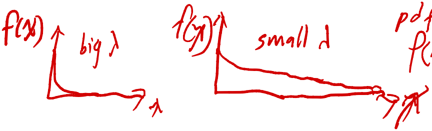
$F^{-1}(u) = \min\{x : F(x) \geq u\}$



Claim still holds: $F^{-1}(u) \leq x \;\Leftrightarrow\; u \leq F(x)$ **by definition**

Exercise: Show that inversion method = naive method for discrete RVs

Markovian GSMPs



pdf
$f(x) = \lambda e^{-\lambda x}$
$E[x] = \frac{1}{\lambda}$

Properties of the Exponential Distribution

If $X \overset{D}{\sim} \exp(\lambda)$ and $Y \overset{D}{\sim} \exp(\mu)$ then

1. $\min(X, Y) \overset{D}{\sim} \exp(\lambda + \mu)$ [indep. of whether min $= X$ or Y]
2. $P(X < Y) = \frac{\lambda}{\lambda+\mu}$
3. $P(X > a + b \mid X > a) = e^{-\lambda b}$ [memoryless property]

Properties 1 and 2 generalize to multiple exponential RVs

Simple GSMP event e'

$F(\cdot\,; s', e', s, E^*) \equiv F(\cdot\,; e')$ and $F_0(\cdot\,; e'; s) \equiv F(\cdot\,; e')$

Markovian GSMPs, Continued

Suppose that all events in a GSMP are simple with exponential clock-setting distn's

Key observation: By memoryless property, whenever GSMP jumps into a state s, clock readings for events in $E(s)$ are mutually independent and exponentially distributed

Simplified Simulation Algorithm (No clock readings needed)

1. (Initialization) Select $s \overset{D}{\sim} \nu$
2. Generate holding time $t^* \overset{D}{\sim} \exp(\lambda)$, where $\lambda = \lambda(s) = \sum_{e_i \in E(s)} \lambda_i$
3. Select $e_i \in E(s)$ as trigger event with probability λ_i / λ
4. Generate the next state $s' \overset{D}{\sim} p(\,\cdot\,; s, e_i)$
5. Set $s = s'$ and go to Step 2

Markovian GSMPs, Continued

S_n = state after n^{th} transition

Structure of a Markovian GSMP

- Sequence $(S_n : n \geq 0)$ is a DTMC with transition matrix $R(s, s') = \sum_{e_i \in E(s)} p(s'; s, e_i)(\lambda_i/\lambda)$
- Given $(S_n : n \geq 0)$, holding times are mutually independent with holding time in $S_n \overset{D}{\sim} \exp(\lambda(S_n))$

Often, occurrence of e_i in s causes state to change to a unique state $y_i = y_i(s)$ with probability 1

Super-Simplified Simulation Algorithm

1. (Initialization) Select $s \overset{D}{\sim} \nu$
2. Generate holding time $t^* \overset{D}{\sim} \exp(\lambda)$, where $\lambda = \sum_{e_i \in E(s)} \lambda_i$
3. Set $s' = y_i(s)$ with probability λ_i/λ
4. Set $s = s'$ and go to Step 2

Markovian GSMPs, Continued

A GSMP $\big(X(t) : t \geq 0)\big)$ with simple, exponential transitions is a continuous-time Markov chain (CTMC) [Ross, Ch. 6]

- Finite or countable state space
- Continuous-time Markov property

$$P\big(X(t+u) = s \mid X(s) : 0 \leq s \leq t\big) = P\big(X(t+u) = s \mid X(t)\big)$$

All CTMCs have foregoing structure

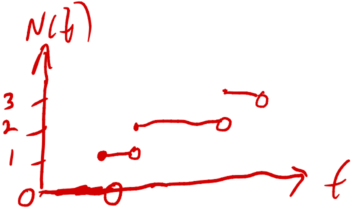
- State sequence is a DTMC
- Holding times mutually independent and $\exp\big(\lambda(s)\big)$ in state s

Q: What can go wrong if events are not simple?

Example of Markovian GSMP: Poisson Process

Definition of Poisson process $\big(N(t) : t \geq 0\big)$ with rate λ

- $S = \{0, 1, 2, \ldots\}$
- Single $\exp(\lambda)$ event
- $p(s+1; s, e) = 1$



Can show that

$$P\big(N(t+s) = m+n \mid N(t) = m\big) = \frac{e^{-\lambda s}(\lambda s)^n}{n!}$$

Poisson dist'n

Examples: # arrivals to a queue, # of lightbulb replacements

Semi-Markovian GSMPs

GSMP $\big(X(t) : t \geq 0\big)$ with simple events such that $|E(s)| = 1$ for all $s \in S$ is a semi-Markov process

Definition of semi-Markov process

- Discrete state space S
- State sequence $(X_n : n \geq 0)$ is a DTMC with transition matrix, say, R
- Holding time in $s \overset{D}{\sim} F(\,\cdot\,; s)$
- "Markov property holds only at state-transition times"

Example: Renewal counting process

- $S = \{0, 1, 2, \ldots\}$
- $R(s, s+1) = 1$ for all $s \in S$
- $F(\,\cdot\,; s) \equiv G(\,\cdot\,)$ for some G