Simulation of Discrete-Time Markov Chains

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CS590M: Simulation
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DTMC Definition

Simplest model for dynamic stochastic system

- $X_n$ = system state after $n$th transition
- $(X_n : n \geq 0)$ satisfies the Markov property

$$P\{X_{n+1} = x \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0\} = P\{X_{n+1} = x \mid X_n = x\}$$

Time-homogeneous DTMC with state space $S$ defined via

1. Transition matrix $P = (P(x, y) : x, y \in S)$, with
   \[ P(x, y) = P\{X_{n+1} = y \mid X_n = x\} \]
2. Initial distribution $\mu = (\mu(x) : x \in S)$, with
   \[ \mu(x) = P\{X_0 = x\} \]

Example: Markovian Jumping Frog

$X_n = \text{lily pad occupied by frog after } n\text{th jump}$

- Frog starts in states 1 and 2 with equal probability

$$P = \begin{bmatrix}
0 & 1/2 & 1/2 \\
1/3 & 0 & 2/3 \\
3/4 & 1/4 & 0
\end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix}
1/2 \\
1/2 \\
0
\end{bmatrix}$$
Computing Probabilities and Expectations

Example: \( \theta = P \{ \text{frog on pad 2 after } k \text{th jump} \} \)

- Write as \( \theta = E[f(X_k)] \), where \( f(x) = \begin{cases} 1 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases} \)
- Sometimes write indicator function \( f(x) \) as \( I(x = 2) \)
- Q: Why is this correct?

Numerical solution for arbitrary function \( f \)

- Q: What is probability distribution after first jump?
- Let \( v_n(i) = P \{ \text{frog on pad } i \text{ after } n \text{th jump} \} \)
  
\[
\begin{bmatrix}
v_n(1) \\ v_n(2) \\ v_n(3)
\end{bmatrix}
\]

- Set \( v_0 = \mu \) and \( v_{m+1}^\top P \) for \( m \geq 0 \)
  
\[
\text{i.e., } v_{m+1}(j) = \sum_{i=1}^{3} v_m(i) P(i,j)
\]
- Then \( E[f(X_k)] = v_k^\top f \), where \( f = [f(1), f(2), f(3)]^\top \)
- Ex: For \( \theta \) as above, take \( f = (0, 1, 0) \) so that \( v_k^\top f = v(2) \)

Simulation of DTMCs

Why simulate?

Naive method for generating a discrete random variable

- Goal: Generate \( Y \) having pmf \( p_i = P\{Y = y_i\} \) and cdf \( c_i = P\{Y \leq y_i\} \) for \( 1 \leq i \leq m \)

\[
\begin{array}{c|c|c}
   i   & p_i & c_i \\
   \hline
   1   & 3/12 & 3/12 \\
   2   & 8/12 & 11/12 \\
   3   & 1/12 & 12/12 \\
\end{array}
\]

- Ex: If \( U = 0.27 \), then return \( Y = y_2 \)
- Q: How can we speed up this algorithm?

Simulation of DTMCs, Continued

Generating a sample path \( X_0, X_1, \ldots \)

1. Generate \( X_0 \) from \( \mu \) and set \( m = 0 \)
2. Generate \( Y \) according to \( P(X_m, \cdot) \) and set \( X_{m+1} = Y \)
3. Set \( m \leftarrow m + 1 \) and go to 2.

Estimating \( \theta = E[f(X_k)] \)

1. Generate \( X_0, X_1, \ldots, X_k \) and set \( Z = f(X_k) \)
2. Repeat \( n \) times to generate \( Z_1, Z_2, \ldots, Z_n \) (i.i.d.)
3. Compute point estimate \( \theta_n = (1/n) \sum_{i=1}^{n} Z_i \)

Can generalize to estimate \( \theta = E[f(W)] \), where \( W = f(X_0, X_1, \ldots, X_k) \)

DTMCs: Recursive Definition

Proposition

- Let \( U_1, U_2, \ldots \) be a sequence of i.i.d. random variables and \( X_0 \) a given random variable
- \((X_n : n \geq 0)\) is a time homogeneous DTMC \( \iff \ X_{n+1} = f(X_n, U_{n+1}) \) for \( n \geq 0 \) and some function \( f \)

In \( \Rightarrow \) direction, \( U_1, U_2, \ldots \) can be taken as uniform

Can use to prove that a given process \((X_n : n \geq 0)\) is a DTMC

Q: What if \( U_0, U_1, \ldots \) are independent but not identical?

Q: Practical advantages of recursive definition?
Example: (s, S) Inventory System

The model
- \( X_n \) = inventory level at the end of period \( n \)
- \( D_n \) = demand in period \( n \)
- If (s, S) policy is followed then

\[
X_{n+1} = \begin{cases} 
X_n - D_{n+1} & \text{if } X_n - D_{n+1} \geq s; \\
S & \text{if } X_n - D_{n+1} < s
\end{cases}
\]

Claim: If \((D_n : n \geq 1)\) is i.i.d. then \((X_n : n \geq 0)\) is a DTMC with state space \(\{s, s+1, \ldots, S\}\)

Q: Critique of model—what might be missing?

Digression: Stationary Distribution of a DTMC

Definition
- Informal: \(\pi\) is a stationary distribution of the DTMC if \(X_n \sim \pi\) implies \(X_{n+1} \sim \pi\)
- Formal:

\[
\pi(j) = P(X_{n+1} = j) = \sum_i P(X_{n+1} = j \mid X_n = i) P(X_n = i) = \sum_i P(i,j)\pi(i) \quad \text{or } \pi^\top = \pi^\top P
\]

- So if \(X_0 \sim \pi\), then \(X_n \sim \pi\) for \(n \geq 1\)

Under appropriate conditions, \(\lim_{n \to \infty} P(X_n = i) = \pi(i)\)
- Also written as \(X_n \Rightarrow X\), where \(X \sim \pi\)

How to estimate \(\theta = \sum_i f(i)\pi(i) = E[f(X)]\) (where \(X \sim \pi\))? 9/13

General State Space Markov Chains: GSSMCs

Problem: With continuous state space,
\(P\{X_{n+1} = x' \mid X_n = x\} = 0!\)
- Solution: Use transition kernel
  \(P(x, A) = P\{X_{n+1} \in A \mid X_n = x\}\)
- In practice: Use recursive definition

Example: Continuous (s, S) inventory system

Example: Random walk on the real line
- Let \(Y_1, Y_2, \ldots\) be an i.i.d. sequence of continuous, real-valued random variables
- Set \(X_0 = 0\) and \(X_{n+1} = X_n + Y_{n+1}\) for \(n \geq 0\)
- Then \((X_n : n \geq 0)\) is a GSSMC with state space \(\mathbb{R}\)

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GSSMC Example: Waiting times in GI/G/1 Queue

The GI/G/1 Queue
- Service center: single server, infinite-capacity waiting room
- Jobs arrive one at a time
- First-come, first served (FCFS) service discipline
- Successive interarrival times are i.i.d.
- Successive service times are i.i.d.

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GI/G/1 Waiting Times, Continued

Notation
- \( W_n \) = the waiting time of the \( n \)th customer (excl. of service)
- \( A_n / D_n \) = arrival/departure time of the \( n \)th customer
- \( V_n \) = processing time of the \( n \)th customer

Recursion (Lindley Equation)
- \( D_n = A_n + W_n + V_n \)
- Thus

\[
W_{n+1} = \left[ D_n - A_{n+1} \right]^+ = \left[ A_n + W_n + V_n - A_{n+1} \right]^+
\]

\[= \left[ W_n + V_n - I_{n+1} \right]^+\]

where \( I_{n+1} = A_{n+1} - A_n \) is \((n + 1)\)st interarrival time and
\([x]^+ = \max(x, 0)\)
- Thus \( (W_n : n \geq 0) \) is a GSSMC
- To simulate: generate the \( V_n \)'s and \( I_n \)'s and apply recursion