Assignment #4 Solutions

1. Truncated distributions are useful if we think we know the general form of an input distribution, but we have some additional information that further restricts the range of possible values.

(a) As stated in the text, the cdf is:

$$F^*(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{F(x) - F(a)}{F(b) - F(a)} & \text{if } a \le x < b \\ 1 & \text{if } x \ge b \end{cases}$$

For the algorithm in Section 8.2.1,

$$\begin{aligned} &P\{X \le x\} = P\{F^{-1}(V) \le x\} = P\{V \le F(x)\} \\ &= P\left\{F(a) + \left(F(b) - F(a)\right)U \le F(x)\right\} \\ &= P\left\{U \le \frac{F(x) - F(a)}{F(b) - F(a)}\right\} \\ &= F^*(x) \end{aligned}$$

(b) For the algorithm of problem 8.3(b),

$$\begin{aligned} &P\{X \le x\} = P\left\{F^{-1}(U) \le x \mid F(a) \le U \le F(b)\right\} \\ &= P\left\{U \le F(x) \mid F(a) \le U \le F(b)\right\} \\ &= \frac{P\left\{U \le F(x), F(a) \le U \le F(b)\right\}}{P\left\{F(a) \le U \le F(b)\right\}} \\ &= \frac{P\left\{U \le F(x), F(a) \le U \le F(b)\right\}}{F(b) - F(a)} \\ &= F^*(x), \end{aligned}$$

since

$$\begin{aligned} &P\left\{U \le F(x), F(a) \le U \le F(b)\right\} \\ &= \begin{cases} 0 & \text{if } x < a \\ P\left\{F(a) \le U \le F(x)\right\} & \text{if } a \le x < b \\ F(b) - F(a) & \text{if } x \ge b. \end{cases} \end{aligned}$$

The first algorithm (pure inversion) has the advantage that only one iteration is required, whereas the second algorithm may require multiple iterations, especially if $F(b) - F(a)$ is small.

(c) Using the inversion method, we have the following algorithm:

Algorithm
1. Generate $U \sim U[0,1]$
2. Return
$$X = \begin{cases} a & \text{if } U < F(a) \\ F^{-1}(U) & \text{if } F(a) \le U < F(b) \\ b & \text{if } U \ge F(b) \end{cases}$$

Equivalently, we can re-express the algorithm as follows.

Algorithm
1. Generate $U : U[0,1]$ and set $Y = F^{-1}(U)$ (equivalently, generate Y from F)
2. Return
$$X = \begin{cases} a & \text{if } Y < a \\ Y & \text{if } a \le Y < b \\ b & \text{if } Y \ge b \end{cases}$$

More concisely, in Step 2 we return $X = \min\big(b, \max(Y,a)\big)$.

2. Geometric random variates.

(a) Using the hint, fix $i \ge 0$ and observe that
$$\begin{aligned} P(X=i) &= P\left(i \le \frac{\ln U}{\ln(1-p)} < i+1 \right) \\ &= P\big(i\ln(1-p) \ge \ln U > (i+1)\ln(1-p)\big) \\ &= P\big(\ln(1-p)^i \ge \ln U > \ln(1-p)^{i+1}\big) \\ &= P\big((1-p)^i \ge U > (1-p)^{i+1}\big) \\ &= (1-p)^i - (1-p)^{i+1} \\ &= (1-p)^i(1-(1-p)) \\ &= (1-p)^i p \end{aligned}$$

To see that this algorithm corresponds to the inversion method, first note that for any non-integer real number y, we have $y = \lfloor y \rfloor + \varepsilon$ for some real number $\varepsilon \in (0,1)$, so that $y-1 = \lfloor y \rfloor + \varepsilon - 1 = \lfloor y \rfloor - \delta$ for some $\delta \in (0,1)$. Thus the smallest integer greater than or equal to $y-1$ is $\lfloor y \rfloor$. As per the extra credit problem, we have, for $x \ge 0$,
$$\begin{aligned} F(x) &= P(X \le x) = P\big(X \le \lfloor x \rfloor\big) \\ &= \sum_{j=0}^{\lfloor x \rfloor} p(1-p)^j = p\sum_{j=0}^{\lfloor x \rfloor} (1-p)^j = p\frac{1-(1-p)^{\lfloor x \rfloor + 1}}{1-(1-p)} \\ &= 1-(1-p)^{\lfloor x \rfloor + 1}, \end{aligned}$$

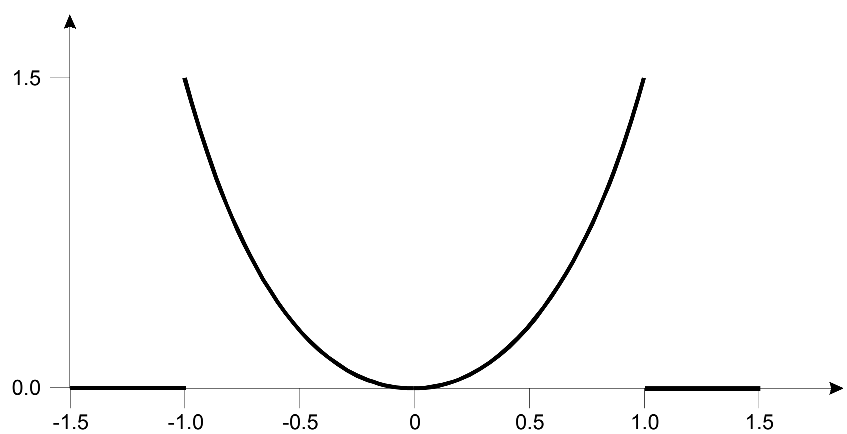
where the second equality follows from the fact that X only takes on integer values, and the fifth equality follows from the standard identity $\sum_{j=0}^{k} x^k = (1-x^{k+1})/(1-x)$. Using the definition of the generalized inverse (needed because of the $\lfloor x \rfloor$ term), we have

$$
\begin{aligned}
F^{-1}(u) &= \min\{x : F(x) \ge u\} \\
&= \min\{x : 1-(1-p)^{\lfloor x \rfloor + 1} \ge u\} \\
&= \min\{x : 1-u \ge (1-p)^{\lfloor x \rfloor + 1}\} \\
&= \min\left\{x : \ln(1-u) \ge \left(\lfloor x \rfloor + 1\right)\ln(1-p)\right\} \\
&= \min\left\{x : \lfloor x \rfloor \ge \frac{\ln(1-u)}{\ln(1-p)} - 1\right\} \\
&= \min\left\{m : m \text{ is an integer and } m \ge \frac{\ln(1-u)}{\ln(1-p)} - 1\right\} \\
&= \left\lfloor \frac{\ln(1-u)}{\ln(1-p)} \right\rfloor,
\end{aligned}
$$

provided that the final ratio of logarithms is non-integer. Here the last equality follows from the previous calculation. The algorithm therefore is, in fact, the inversion method, since $\ln(1-U)/\ln(1-p)$ is non-integer with probability 1 for a uniform random number U.

(b) Because the uniforms in the given algorithm are mutually independent, steps 2 and 3 constitute a sequence of Bernoulli trials with success probability $P(U \le p) = p$. The variable i is incremented at, and only at, each "failure", so that X simply counts the number of trials until the first success. As discussed in class, X therefore has a geometric distribution.

3. The density looks as follows



(a) **Inversion**: $F(x) = 0.5(x^3 + 1)$ for $-1 \le x \le 1$, so inversion yields the formula $X = (2U-1)^{1/3}$. (Note that, in general, there will be two complex cube roots and one real-valued cube root. We obviously want to take the real-valued cube root.)

(b) **Composition**: Write $f(x) = 0.5 \cdot 1_{[-1,0)}(x) \cdot 3x^2 + 0.5 \cdot 1_{[0,1]}(x) \cdot 3x^2$, so that $F(x) = 0.5 \cdot 1_{[-1,0)}(x) \cdot (x^3+1) + 0.5 \cdot 1_{[0,1]}(x) x^3$. We can use inversion for each part. The algorithm is

1. Generate U_1, U_2 iid $U[0,1]$.
2. If $U_1 \le 0.5$, then return $(U_2 - 1)^{1/3}$ [equivalently, $(-U_2)^{1/3}$], else return $U_2^{1/3}$.

(c) **Acceptance/rejection**: take g as a uniform distribution on [-1,1], i.e, $g(x) = 0.5 \cdot 1_{[-1,1]}(x)$, and take $c = \sup_x f(x)/g(x) = 3$. The algorithm is

1. Generate U_1, U_2 iid $U[0,1]$.
2. Set $Y = 2U_1 - 1$
3. If $U_2 \le Y^2$, then return $X = Y$, else go to Step 1.

Inversion and composition each require a cube-root evaluation, and inversion requires one less uniform random number, so inversion is better than composition. The A/R algorithm avoids the cube root operation, but requires $c = 3$ pairs (U_1, U_2), i.e., 6 uniforms random numbers, on average, so whether A/R is better or worse than inversion depends on the relative cost of cube root versus pseudorandom number generation.

4. First note that $H(t) = t^a$. Given that $T_{n-1} = y$, the cdf of T_n is $F(x) = 1 - e^{-(H(x) - H(y))}$ for $x \ge y$, so

$$F^{-1}(u) = H^{-1}\left(H(y) - \ln(1-u)\right) = \left(H(y) - \ln(1-u)\right)^{1/a} = \left(y^a - \ln(1-u)\right)^{1/a}$$

since $H^{-1}(t) = t^{1/a}$. We can replace $1-u$ with u in the usual way to obtain:

<u>Algorithm</u>

1. Set $T_0 = 0$ and $n = 1$
2. Generate U : $U[0,1]$
3. Set $T_n = \left(T_{n-1}^a - \ln(U)\right)^{1/a}$
4. Set $n \leftarrow n+1$ and go to Step 2

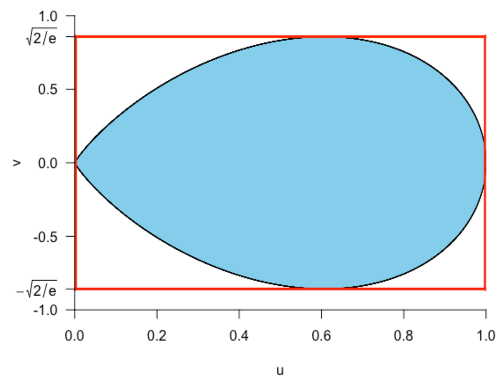
6. Ratio-of-Uniforms method.

(a) Following the suggestion, we have $pf(x) = e^{-x^2/2}$ and $\sqrt{pf(x)} = e^{-x^2/4}$. Clearly, $u^* = \max_x \sqrt{pf(x)} = \max_x pf(x) = 1$ (i.e., when $x = 0$). To compute v_* and v^*, observe that

$$\frac{d}{dx} x\sqrt{pf(x)} = \frac{d}{dx} x e^{-x^2/4} = e^{-x^2/4}\left(1 - \frac{x^2}{2}\right)$$

Setting the derivative equal to 0, we see that the maximum and minimum values are achieved at $x^* = \pm\sqrt{2}$, and the values themselves are $x^*\sqrt{pf(x^*)} = \pm\sqrt{2/e}$. Hence $v_* = -\sqrt{2/e}$ and $v^* = \sqrt{2/e}$.

(b) Following the hint, we have $u = \sqrt{pf(v/u)} = e^{-v^2/4u^2}$ and, solving for v, we get $v = \pm 2u\sqrt{-\ln u}$. So the plot of the region $S = \left\{(u,v) : u \le \sqrt{pf(v/u)}\right\}$ and the bounding rectangle (in red) is as follows:

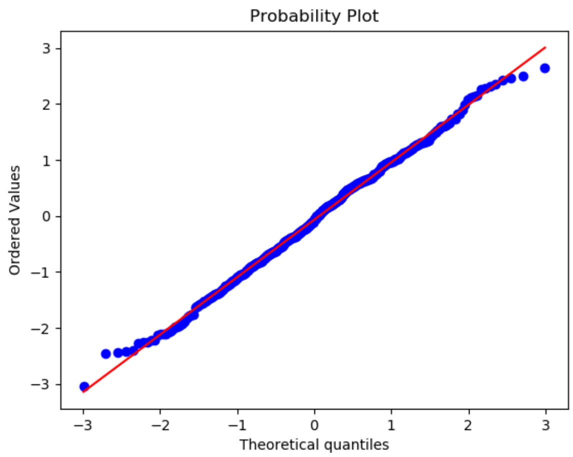


(c) Putting the above results together, the final algorithm is as follows:

1. Generate independent uniform numbers U_1 and U_2
2. Set $U = U_1$ and $V = \sqrt{2/e}(2U_2 - 1)$
3. Set $Z = V/U$
4. If $U^2 \le e^{-Z^2/2}$ return Z, else go to step 1.

(d) The acceptance probability is $\alpha = \dfrac{p/2}{u^*(v^* - v_*)} = \dfrac{\sqrt{\pi e}}{4} \approx 0.73$. As for the ordinary acceptance-rejection method the expected number of rounds is $1/\alpha$, so the expected number of uniform variates is $2/\alpha \approx 2.74$. (Professional-grade generators try to improve performance by more tightly bounding S using, e.g., polygons or ellipsoids.)

(e) See the website for example Python code. Our Q-Q plot is as follows:


Probability Plot
Ordered Values
Theoretical quantiles
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3