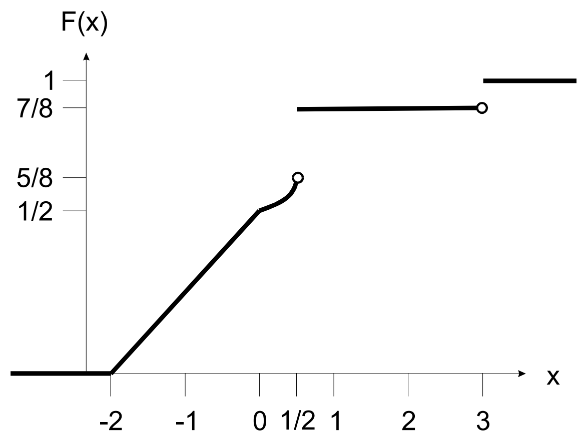
Assignment #2 Solutions

1. The distribution function looks roughly as follows:



(a) We have

$$F^{-1}(u) = \begin{cases} 4u-2 & \text{for } 0 \le u \le 1/2 \\ \sqrt{2u-1} & \text{for } 1/2 < u \le 5/8 \\ 1/2 & \text{for } 5/8 < u \le 7/8 \\ 3 & \text{for } 7/8 < u \le 1 \end{cases}$$

Thus, the algorithm is as follows:

Algorithm

1.Generate a uniform(0,1) random variable U
2. Return X, where

$$X = \begin{cases} 4U-2 & \text{if } 0 \le U \le 1/2 \\ \sqrt{2U-1} & \text{if } 1/2 < U \le 5/8 \\ 1/2 & \text{if } 5/8 < U \le 7/8 \\ 3 & \text{if } 7/8 < U \le 1 \end{cases}.$$

(b) We have

$$\begin{aligned} E[X] &= \int_{-2}^{0} x f(x)\,dx + \int_{0}^{1/2} x f(x)\,dx + (1/2)(2/8) + (3)(1/8) \\ &= \int_{-2}^{0} \frac{x}{4}\,dx + \int_{0}^{1/2} x^2\,dx + (1/8) + (3/8) \\ &= \left.\frac{x^2}{8}\right|_{-2}^{0} + \left.\frac{x^3}{3}\right|_{0}^{1/2} + \frac{1}{2} \\ &= -\tfrac{1}{2} + \tfrac{1}{24} + \tfrac{1}{2} = \tfrac{1}{24} \end{aligned}$$

and

$$
\begin{aligned}
E[X^2] &= \int_{-2}^{0} x^2 f(x)\,dx + \int_{0}^{1/2} x^2 f(x)\,dx + (1/4)(2/8) + (9)(1/8) \\
&= \int_{-2}^{0} \frac{x^2}{4}\,dx + \int_{0}^{1/2} x^3\,dx + \frac{19}{16} \\
&= \left.\frac{x^3}{12}\right|_{-2}^{0} + \left.\frac{x^4}{4}\right|_{0}^{1/2} + \frac{19}{16} \\
&= \tfrac{8}{12} + \tfrac{1}{64} + \tfrac{19}{16} = \tfrac{359}{192}
\end{aligned}
$$

So $\mathrm{Var}[X] = E[X^2] - E^2[X] = \tfrac{359}{192} - \left(\tfrac{1}{24}\right)^2 = \dfrac{269}{144} \approx 1.868$.

(c) Note that the assumptions on F are necessary for the result to hold; e.g., if X is a discrete random variable, then $F(X)$ cannot possibly have a uniform(0,1) distribution. To prove the result, fix $y \in [0,1]$. Then

$$P\{F(X) \le y\} = P\{F^{-1}\left(F(X)\right) \le F^{-1}(y)\} = P\{X \le F^{-1}(y)\} = F\left(F^{-1}(y)\right) = y$$

and hence U has a uniform cdf. The first equality holds because F^{-1} is nondecreasing, the third equality holds by definition of F, and the second and fourth equalities hold from the properties given in the hint.

2. Hospital unit

(a) GSMP building blocks:

<u>State space</u>: $S = \{0,1\}^{b_1+b_2}$

<u>Active events</u>: for $s = (m_1,\ldots,m_{b_1},n_1,\ldots,n_{b_2}) \in S$,

$e_0 \in E(s)$ always
$e_{1,i} \in E(s)$ iff $m_i = 1$
$e_{2,i} \in E(s)$ iff $n_i = 1$
$e_{3,i} \in E(s)$ iff $m_i = 1$

<u>State-transition probabilities</u>: As in the hint, define $i_{\mathrm{SCU}}(s) = \min\{i : m_i = 0\}$ with $i_{\mathrm{SCU}}(s) = -1$ if $\min_{1\le i\le b_1} m_i = 1$, and similarly define $i_{\mathrm{ICU}}(s) = \min\{i : n_i = 0\}$ with $i_{\mathrm{ICU}}(s) = -1$ if $\min_{1\le i\le b_1} n_i = 1$. Then, for $s = (m_1,\ldots,m_{b_1},n_1,\ldots,n_{b_2}), s' = (m_1',\ldots,m_{b_1}',n_1',\ldots,n_{b_2}') \in S$

for $e^* = e_0$,
if $i_{\mathrm{SCU}}(s) \ne -1$ and $i_{\mathrm{ICU}}(s) \ne -1$, then
$p(s';s,e^*) = p$ when $s' = s$ except that $m_{i_{\mathrm{SCU}}(s)}' = 1$
and
$p(s';s,e^*) = 1-p$ when $s' = s$ except that $n_{i_{\mathrm{ICU}}(s)}' = 1$
if $i_{\mathrm{SCU}}(s) = -1$ and $i_{\mathrm{ICU}}(s) \ne -1$, then
$p(s';s,e^*) = p$ when $s' = s$
and
$p(s';s,e^*) = 1-p$ when $s' = s$ except that $n_{i_{\mathrm{ICU}}(s)}' = 1$

if $i_{\text{SCU}}(s) \neq -1$ and $i_{\text{ICU}}(s) = -1$, then

$p(s';s,e^*) = p$ when $s' = s$ except that $m'_{i_{\text{SCU}}(s)} = 1$

and

$p(s';s,e^*) = 1 - p$ when $s' = s$

if $i_{\text{SCU}}(s) = i_{\text{ICU}}(s) = -1$, then

$p(s';s,e^*) = 1$ when $s' = s$

for $e^* = e_{1,i}$,

$p(s';s,e^*) = 1$ when $s' = s$ except that $m'_i = 0$

for $e^* = e_{2,i}$,

$p(s';s,e^*) = 1$ when $s' = s$ except that $n'_i = 0$

for $e^* = e_{3,i}$,

$p(s';s,e^*) = 1$ when $s' = s$ except that (1) $m'_i = 0$ and (2) if $i_{\text{ICU}}(s) \neq -1$, then $n'_{i_{\text{ICU}}(s)} = 1$

$p(s';s,e^*) = 0$ otherwise

<u>Speeds</u>: $r(s,e) = 1$ for all $s \in S$ and $e \in E(s)$

<u>Clock-setting distributions</u>:

$F(x;s',e_0,s,e^*) \equiv F(x;e_0) = P(A \le x)$

$F(x;s',e_{1,i},s,e^*) \equiv F(x;e_{1,i}) = P(L_{\text{SCU}} \le x)$ for $1 \le i \le b_1$

$F(x;s',e_{2,i},s,e^*) \equiv F(x;e_{2,i}) = P(L_{\text{ICU}} \le x)$ for $1 \le i \le b_2$

$F(x;s',e_{3,i},s,e^*) \equiv F(x;e_{3,i}) = P(Q \le x)$ for $1 \le i \le b_1$

<u>Initial distribution</u>: $\nu(s_0) = 1$, where $s_0 = (0,0,\ldots,0)$. Initial clock-setting distribution for e_0 is as above.

(b) For convenience, set $h(s) = \sum_{i=1}^{b_1} m_i + \sum_{i=1}^{b_2} n_i$ for $s = (m_1,\ldots,m_{b_1},n_1,\ldots n_{b_2}) \in S$. I.e., $h(s)$ is the number of occupied beds in state *s*.

(i) Performance measure = $E[T]$, where $T = \min\{t \ge 0 : X(t) = (1,1,\ldots,1)\}$.

(ii) Performance measure = $E\left[\frac{1}{30}\int_0^{30} I\left(b_1 + b_2 - h\left(X(t)\right) \ge 7\right) dt\right]$,.

(iii) Performance measure = $P\{T \le 55\}$, where *T* is defined as in (i).

(iv) Performance measure = $E[N_{\text{Trans}} / N_{\text{Arr}}]$, where

$N_{\text{Arr}} = \sum_{n=1}^{\infty} I\left(\zeta_n \le 30 \text{ and } E^*(S_{n-1},C_{n-1}) = \{e_0\}\right)$ and

$N_{\text{Trans}} = \sum_{n=1}^{\infty} I\left(\zeta_n \le 30 \text{ and } E^*(S_{n-1},C_{n-1}) = \{e_0\} \text{ and } h(S_n) = h(S_{n-1})\right)$. Take $0/0 = 0$.

(c) See the class web page for a Python program that solves the problem. Note that, to generate samples of *A*, we can use inversion: $F(x;a) = (x/a)^2$, so that $F^{-1}(u) = a\sqrt{u}$. To generate samples of L_{SCU}, observe that $E[L_{\text{SCU}}] = 2/\lambda = 0.25$, so that $\lambda = 8$, and hence we can generate a sample as $-(\log U_1 + \log U_2)/8$, where U_1, U_2 are independent uniform random numbers. Samples of L_{ICU} are generated in the same manner. Finally, to generate a sample of Q, we can generate a triangular random variable on [0, 2] using the result in Assignment #1, and then shift and scale to get the desired result, i.e., we can generate a sample as $0.5(U_1 + U_2) + 0.5$. A tricky aspect of the coding is the fact that events must be canceled at an SCU departure or at a critical event. Our results were as follows. (All times are in days.)

Performance measure	# reps	Point Estimate	99% CI	CI Half-width
E[Time to first fill-up]	70000	30.44	[30.15, 30.72]	0.29
E[fraction of time with ≥ 7 empty beds]	6500	0.07	[0.0706, 0.0718]	0.0006
P{ Time to first fill-up ≤ 55}	15000	0.83	[0.8239, 0.8397]	0.0079
E[fraction of transferred arrivals]	3000	0.05	[0.0533, 0.0544]	0.0005

3. Markov-chain Monte Carlo (MCMC).

(a) When $\pi(j)Q(j,i) > \pi(i)Q(i,j)$, we have $\alpha(i,j) = 1$ and

$$\alpha(j,i) = \frac{\pi(i)Q(i,j)}{\pi(j)Q(j,i)} < 1,$$

so that $\pi(j)Q(j,i)\alpha(j,i) = \pi(i)Q(i,j)\cdot 1 = \pi(i)Q(i,j)\alpha(i,j)$. An almost identical argument holds when $\pi(j)Q(j,i) \le \pi(i)Q(i,j)$.

(b) For $i \ne j$, we have

$$\begin{aligned} P(i,j) &= P\{X_{n+1} = j \mid X_n = i\} \\ &= P\{Y = j, U \le \alpha(i,j) \mid X_n = i\} \\ &= P\{Y = j \mid X_n = i\}P\{U \le \alpha(i,j)\} \quad \text{[by independence]} \\ &= Q(i,j)\alpha(i,j) \quad \text{[by definition of the algorithm and properties of U(0,1)]} \end{aligned}$$

By part (a), we have, for $i \ne j$,

$$\pi(i)P(i,j) = \pi(i)Q(i,j)\alpha(i,j) = \pi(j)Q(j,i)\alpha(j,i) = \pi(j)P(j,i)$$

The outer expressions are trivially equal when $i = j$, so we have shown that $\pi(i)P(i,j) = \pi(j)P(j,i)$ for all i and j. Now sum over i and use the fact that $P(j,\cdot)$ is a pmf for each j to obtain

$$\sum_i \pi(i)P(i,j) = \sum_i \pi(j)P(j,i) = \pi(j)\sum_i P(j,i) = \pi(j),$$

So, by definition, π is a stationary distribution for $\{X_n : n \ge 0\}$.

(c) Let ϕ be the pdf of W, and define

$$h(i) = \begin{cases} i^{-2}e^{\cos(i)} & \text{if } i \ne 0 \\ 1 & \text{if } i = 0 \end{cases}$$

so that $\pi = \theta h$. Observe that $Q(i,j) = Q(j,i) = \phi(|i-j|)$ for all i and j, so that we can take

$$\alpha(i,j) = \min\left(1, \frac{\pi(j)}{\pi(i)}\right) = \min\left(1, \frac{h(j)}{h(i)}\right).$$

Then the algorithm is as follows:

1. Set $X_0 = 0, Z = 0, m = 0$
2. Generate $V : U[0,1]$ and set $W = \lfloor (2k+1)V - k \rfloor$
3. Set $Y = X_m + W$
4. Generate $U : U[0,1]$
5. If $U \le \min\left(1, h(Y)/h(X_m)\right)$, then set $X_{m+1} = Y$, else set $X_{m+1} = X_m$
6. Set $Z \leftarrow Z + g\left(X_{m+1}\right)$ and $m \leftarrow m+1$
7. If $m = n$, then return $Z/(n+1)$ as a point estimate, else go to Step 2

By the cited SLLN, $Z/(n+1) \to S$ with probability 1 as $n \to \infty$, so the estimator is strongly consistent. Note that if $h(Y) \ge h(X_m)$, then in Step 5 we can immediately set $X_{m+1} = Y$ without having to generate *U*. The general algorithm is called the Hastings-Metropolis algorithm, and the special case in which $Y = X_m + W$ is called the Metropolis Random-Walk algorithm. These algorithms can be extended to the setting of GSSMCs, in which case they are used to estimate integrals rather than sums. This class of algorithms has proven very powerful in practice, especially when a state is a high-dimensional vector (so that the sum or integral is multidimensional). Then the Hastings-Metropolis iteration can, for example, be applied to one dimension at a time, e.g., in a round-robin manner; the Gibbs sampler is a well-known example of such a method. MCMC algorithms have revolutionized the area of Bayesian statistical models, permitting for the first time the computation of complicated "posterior" expectations that correspond precisely to the problem of estimating sums and integrals of the type discussed above. (In this setting, the "posterior" probability distribution is usually high-dimensional and known only up to a multiplicative constant and, as in the homework problem, this constant is difficult or essentially impossible to compute. The key feature of the algorithm is that it uses only *ratios* of the distribution π, so that the constant cancels out, and thus does not need to be known a priori.)

4. Analytical solution of gambling game.

(a) Let $C_n = 1$ if the *n*th coin flip is heads and $C_n = -1$ if the *n*th coin flip is tails. Then $X_{n+1} = g(X_n, C_{n+1})$, where $g(x,c) = x + c$, so the X_n 's satisfy a recursion as in Lecture 2, Slide 8, so that $\{X_n : n \ge 0\}$ is a Markov chain.

(b) The initial distribution is $\mu(0) = 1$ and $\mu(i) = 0$ for $i \ne 0$. The transition matrix is

	0	1	2	3
0	0	1	0	0
1	0.5	0	0.5	0
2	0	0.5	0	0.5
3	0	0	0	1

(c) Write $y_i = E_i[L]$ for $i = 0,1,2,3$. Then the first-step decomposition equations are

(1) $y_0 = 1 + y_1$

(2) $y_1 = 1 + 0.5 y_0 + 0.5 y_2$

(3) $y_2 = 1 + 0.5 y_1$

Substitute (1) and (3) into (2) to get

$$y_1 = 1 + 0.5(1 + y_1) + 0.5(1 + 0.5) y_1 = 2 + 0.75 y_1$$

so that $y_1 = 8$ and, from (1), $E[L] = y_0 = 9$. Hence the expected gain is $8.99 - 9 = -0.01$.