Solutions to Sample Midterm Questions

1. Parameter fitting.

(a) Density and likelihood function

(i) The density is $f_X(x) = \frac{d}{dx} F_X(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{-(a+1)} = ab^a x^{-(a+1)}$ for $x \ge b$ and

$f_X(x) = 0$ for $x < b$

(ii) The likelihood function is

$$L_n(a,b;X_1,\ldots,X_n) = \prod_{i=1}^{n} f_X(X_i)$$
$$= \prod_{i=1}^{n} I(b \le X_i) \times a^n b^{na} \prod_{i=1}^{n} X_i^{-(a+1)}$$

where I is an indicator function.

(b) Since L_n is initially increasing in *b*, choose *b* as large as possible: $\hat{b} = X_{(1)} = \min_{1 \le i \le n} X_i$.

To choose $\hat{a}$, maximize the function $h(a) = a^n \hat{b}^{na} \prod_{i=1}^{n} X_i^{-(a+1)}$. Equivalently, maximize

$\tilde{h}(a) = \log h(a) = n \log a + na \log \hat{b} - (a+1)\sum_{i=1}^{n} \log X_i$.

$\frac{d}{da}\tilde{h}(a) = \frac{n}{a} + n \log \hat{b} - \sum_{i=1}^{n} \log X_i = 0$ implies $\hat{a} = \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{X_i}{X_{(1)}}\right)\right]^{-1}$.

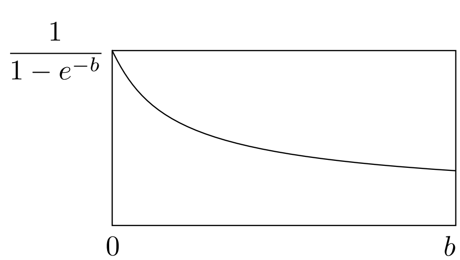
(c) Given $\hat{b}$ we have

$$E[X] = \int_b^\infty x f_X(x)\,dx = \int_b^\infty a\hat{b}^a x^{-a}\,dx = \left.\frac{a\hat{b}^a x^{1-a}}{1-a}\right|_b^\infty = \frac{a\hat{b}}{a-1} .$$

Substitute the sample mean $\bar{X}_n$ for $E[X]$ and solve the equation $\bar{X}_n = \frac{a\hat{b}}{a-1}$ to get $\hat{a} = \frac{\bar{X}_n}{\bar{X}_n - \hat{b}}$.

2. Random variate generation

(a) The picture is as follows.



Using the basic version of the acceptance-rejection method, we obtain the following algorithm:

Algorithm:

1. Generate uniform random numbers U_1, U_2
2. Set $Z_1 = bU_1$ and $Z_2 = U_2/(1-e^{-b})$.
3. If $Z_2 \le \dfrac{e^{-Z_1}}{1-e^{-b}}$, then return Z_1 , else go to Step 1.

(b) A much more efficient inversion algorithm is as follows, based on the fact that $F_X(x) = \dfrac{1-e^{-x}}{1-e^{-b}}$:

Algorithm:

1. Generate a uniform random number U
2. Return $X = -\ln\left(1-(1-e^{-b})U\right)$

3. Linear congruential generators

a) Starting with $x_0 = 1$, we have the following results

i	x_i	$3^i \bmod 7$
1	3	3
2	2	2
3	6	6
4	4	4
5	5	5
6	1	1

By inspection of the second column, we see that all values between 1 and 6 occur, so the generator has full period. Inspection of the third column shows that 3 is a primitive element modulo 7, which also proves that the generator has full cycle since 7 is prime.

b) From Lecture 6, Slide 8, the RANDU generator is an MCG with modulus 2^{31}. By the Claim on Slide 10, the period is at most $2^{k-2} = 2^{6-2} = 2^4 = 16$.

4. Bayesian estimation. Write $X_{(n)} = \max_{1\le i\le n} X_i$, and note that $X_i \ge 0, \forall i$. Then the likelihood is given by $f(X_{1:n} \mid a) = \prod_{i=1}^n [I(X_i \le a)/a] = I(X_{(n)} \le a)/a^n$, where $I(A)$ denotes the indicator of event A and $X_{1:n} = X_1,\ldots,X_n$. The prior is $f(a) = I(0 \le a \le b)/b$. By Bayes' Theorem, the posterior density of a is

$$f(a \mid X_{1:n}) = \frac{f(X_{1:n} \mid a)f(a)}{\int f(X_{1:n} \mid a)f(a)\,da} = \frac{I(X_{(n)} \le a)I(0 \le a \le b)a^{-n}b^{-1}}{\int I(X_{(n)} \le a)I(a \le b)a^{-n}b^{-1}\,da} = \frac{I(X_{(n)} \le a \le b)a^{-n}}{\int I(X_{(n)} \le a \le b)a^{-n}\,da}$$

$$= \frac{I(X_{(n)} \le a \le b)a^{-n}}{\int_{X_{(n)}}^{b} a^{-n}\,da} = \frac{I(X_{(n)} \le a \le b)a^{-n}}{(X_{(n)}^{1-n} - b^{1-n})/(n-1)}.$$

Thus the posterior mean is $\hat{a} = \int_a af(a \mid X_{1:n})\,da = \dfrac{\int_{X_{(n)}}^{b} a^{1-n}\,da}{(X_{(n)}^{1-n} - b^{1-n})/(n-1)} = \dfrac{(X_{(n)}^{2-n} - b^{2-n})/(n-2)}{(X_{(n)}^{1-n} - b^{1-n})/(n-1)}$.