Sample Midterm Questions

1. Suppose that we have observed data values $X_1,\ldots,X_n$ and we wish to fit a cdf of the following form:

$$F_X(x)=\begin{cases} 1-\left(\dfrac{x}{b}\right)^{-a} & \text{if } x\ge b; \\ 0 & \text{if } x<b, \end{cases}$$

where $a,b>0$. (Parameter a determines the "shape" and parameter b the support region.)

a) Write down the density function $f_X(x)$ and then the likelihood function $L_n(a,b;X_1,\ldots,X_n)$.

b) Derive formulas for $\hat{a}$ and $\hat{b}$, the maximum likelihood estimates for the unknown parameters a and b. [Hint: First determine $\hat{b}$ and then maximize the function $h(a)=L_n(a,\hat{b};X_1,\ldots,X_n)$ to determine $\hat{a}$.

c) Suppose that you have computed a maximum likelihood estimator $\hat{b}$ of b as in part (b). Give the formula for a method-of-moments estimator of a under the assumption that $a>1$.

2. Consider the truncated exponential distribution given by $f_X(x)=\dfrac{e^{-x}}{1-e^{-b}}I(0\le x\le b)$.

a) Give an algorithm for generating a sample from $f_X(x)$ based on acceptance-rejection with a uniform majorizing density.

b) Give an algorithm for generating a sample from $f_X(x)$ based on inversion.

3. Linear congruential generators

a. Does the linear congruential generator $x_{n+1}=3x_n \bmod 7$ have full period? Justify your answer.

b. If we extract the lowest 6 bits from each of a sequence of seeds produced by the RANDU generator, what is the maximum period of the resulting sequence?

4. Suppose that we are given i.i.d. observations $X_1,X_2,\ldots,X_n$ from a Uniform$[0,a]$ distribution, and we want to estimate the parameter a using Bayesian methods. Suppose that our prior distribution on a is Uniform$[0,b]$. Give a formula for the posterior-mean estimator $\hat{a}$.