Sampling, Sketching, Streaming, Small-Space Optimization: Algorithmic Approaches for Analyzing Large Graphs

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• **Classic Big Graphs**
  
  Call graph, web graph, IP graph, social networks, citation networks, protein interaction and metabolic networks....

  **Challenge:** Can’t use conventional algorithms on graphs this large. Often can’t even store graph in memory. Graphs may be changing over time and data may be distributed.
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  Gives a natural way to encode structural information when there’s data about both basic entities and their relationships.
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• Want streaming, parallel, distributed algorithms…
Tutorial Goals and Caveats

Present some new algorithmic primitives for large graphs.
Techniques are widely applicable; we’ll be platform agnostic.
Won’t be comprehensive; will cherry pick illustrative results.
Focus on arbitrary graphs rather than specific applications.
Won’t focus on proofs but will give basic outline when it helps convey why certain approaches are effective.
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• **Resources**

Survey: SIGMOD Record

Tutorial: Slides and Bibliography
http://people.cs.umass.edu/~mcgregor/graphs

Lectures: Ten Lectures on Graph Streams
https://people.cs.umass.edu/~mcgregor/courses/CS711S18/
Overview
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- **Part I: Sampling**  Sampling for finding densest subgraphs, small matchings, triangles, spectral properties.

  “Different sampling techniques for different problems”
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• **Part II: Sketching**  Dimensionality reduction for graph data. Examples include connectivity and sparsification.
  
  “Homomorphic compression: sketch first and then run algorithms on the sketched data”
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• **Part IV: Small-Space Optimization**  Combining sparsification and multiplicative weights for fast, small-space optimization. Examples include large matching and correlation clustering.
Recurring Theme
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What’s appropriate notion of lossy compression for graphs?

- If compression is easy, we get faster and more-space efficient algorithms by using existing algorithms on compressed graphs.
Part 1

Sampling

Uniform Sampling + Densest Subgraph
Snape Sampling + Matching
Monochromatic Sampling + Clustering Coefficient
Edge-Weighted Sampling + Cuts and Sparsification
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See tutorial Gionis, Tsourakakis [KDD 15]
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**Thm** Sample of $\tilde{O}(\varepsilon^{-2} n)$ edges uniformly and find the densest subgraph in sampled graph. Gives a $(1 + \varepsilon)$-approx whp.

McGregor et al. [MFCS 15], Esfandiari et al. [SPAA 16]

Mitzenmacher et al. [KDD 15]
Why Uniform Sampling Works...
We're essentially sampling each edge w/p $p \approx \varepsilon^{-2n/m}$.
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So max density of sampled graph gives $1+\varepsilon$ approx.
Part I

Sampling

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Monochromatic Sampling + Clustering Coefficient
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• **Matching Problem** Find large set of edges such that no two edges share an endpoint.
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\[ n \gg k/2 \]
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• **Theorem** If \( G \) has max matching size \( k \), then \( O(k^2 \log k) \) SNAPE samples will include a max matching from \( G \).

  *Chitnis et al. [SODA 16], Bury, Schwiegelshohn [ESA 15]*
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Take $O(k^2 \log k)$ samples; apply analysis to all edges.
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• **Thm** Can additively estimate $\kappa$ from $\tilde{O}(\sqrt{n})$ samples.

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• **Proof Idea** Compute expectation and variance of number of triangles amongst sampled edges and apply Chebyshev bound.
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• **Simpler Thm** If min-cut is $\gg \varepsilon^{-2}\log n$ then $p_e=1/2$ works.
Proof Idea of Simpler Theorem ...
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- **Lemma (Karger)** The number of cuts with $k$ edges is $< \exp(2k \log n / \lambda)$ where $\lambda$ is size of min-cut.

Result then follows by substituting bound for $\lambda$ and applying union bound over all cuts.
Part II

Sketching

What is sketching?
Surprising connectivity example
Revisiting graph cuts and sparsification
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• **Example “\(l_0\) Sampling” Sketch** Can be used to sample uniformly from non-zero entries of the vector where \( D = \text{polylog}(N) \).

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- **Question** What about analyzing massive graphs via sketches?
Part II
Sketching

What is sketching?

Surprising connectivity example

Revisiting graph cuts and sparsification
• **Communication Problem** \( n \) players each have a list of their friends. Simultaneously, they each send a message to a central player who deduces if underlying graph is connected.
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• **Thm** $O(\text{polylog } n)$ bit message from each player suffices.

*Ahn, Guha, McGregor* [SODA 12]
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  - Central player needs to know about the special friendship.
  - Participant doesn’t know which friendships are special.
  - Participants may have \(\Omega(n)\) friends.
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- Algorithm (Spanning Forest):
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- Algorithm (Spanning Forest):
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Ingredient 1: Basic Algorithm

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- **Algorithm (Spanning Forest):**
  - For each node: pick incident edge
  - For each connected comp: pick incident edge
  - Repeat until no edges between connected comp.

- **Lemma** After $O(\log n)$ rounds selected edges include spanning forest.
Ingredient 2: Sketching Neighborhoods
For node $i$, let $a_i$ be vector indexed by node pairs. Non-zero entries: $a_{i}[i,j]=1$ if $j>i$ and $a_{i}[i,j]=-1$ if $j<i$.

$\{1,2\}$  $\{1,3\}$  $\{1,4\}$  $\{1,5\}$  $\{2,3\}$  $\{2,4\}$  $\{2,5\}$  $\{3,4\}$  $\{3,5\}$  $\{4,5\}$

$a_1 = (1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$
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$a_1 = (1,1,0,0,0,0,0,0,0,0,0,0)$

$a_2 = (-1,0,0,0,1,0,0,0,0,0,0,0)$
Ingredient 2: Sketching Neighborhoods

For node $i$, let $a_i$ be vector indexed by node pairs. Non-zero entries: $a_i[i,j]=1$ if $j>i$ and $a_i[i,j]=-1$ if $j<i$.

\[
\begin{align*}
a_1 &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
a_2 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
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Ingredient 2: Sketching Neighborhoods

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$$a_1 + a_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
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**Lemma** For any subset of nodes $S \subset V$, non-zero entries of $\sum_{j \in S} a_j$ are edges across cut $(S,V \setminus S)$.
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**Lemma** For any subset of nodes $S \subset V$, non-zero entries of $\sum_{j \in S} a_j$ are edges across cut $(S, V \setminus S)$

**Player j** sends $M(a_j)$ where $M$ is “$l_0$ sampling” sketch.
Recipe: Sketch & Compute on Sketches
Recipe: Sketch & Compute on Sketches

- Player with Address Books: Player_j sends M_{aj}
Recipe: Sketch & Compute on Sketches

- Player with Address Books: Player $j$ sends $M_{aj}$
- Central Player: “Runs Algorithm in Sketch Space”
Recipe: Sketch & Compute on Sketches

- **Player with Address Books:** Player \( j \) sends \( M_a_j \)
- **Central Player:** “Runs Algorithm in Sketch Space”
- Use \( M_a_j \) to get incident edge on each node \( j \)
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  - For $i=2$ to $\log n$:
    - To get incident edge on component $S \subset V$ use:
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      \[
      \sum_{j \in S} M_{aj} = M\left(\sum_{j \in S} a_j\right)
      \]
Recipe: Sketch & Compute on Sketches

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Recipe: Sketch & Compute on Sketches

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\[
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\]

**Detail:** Actually each player sends $\log n$ independent sketches $M_1a_j, M_2a_j, \ldots$ and central player uses $M_ia_j$ when emulating $i^{th}$ iteration of the algorithm.
• **Thm** $O(\text{polylog } n)$ bit message from each player suffices.
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• **Various extensions** For example, with $\tilde{O}(k)$ bit messages, can find all edges that participate in cuts of size less than $k$. 
Part II

Sketching

What is sketching?
Surprising connectivity example
Revisiting graph cuts and sparsification
• **Thm** $O(\varepsilon^{-2} \text{polylog } n)$ bit messages suffice for central player to construct sparsifier and approx all graph cuts.

  *Guha, McGregor, Tench [PODS 15], Kapralov et al. [STOC 14]*
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• **Main Ideas**

  1. For a graph G, can find all edges in small cuts.
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• **Main Ideas**
  1. For a graph $G$, can find all edges in small cuts.
  2. For large cuts, suffices to sample edges with prob. $1/2$. 
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Part III

Streaming

Revisiting Matching
Correlation Clustering
Coloring Graphs
Coverage and Submodular Maximization
• **Two Main Graph Stream Models**
  
  • *Insert-Only Model*: Input is a stream of edges.
  
  • *Insert-Delete Model*: Edge insertions and edge deletions.
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Mark and Erica are now friends.

Like · Add Friend
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- **Insert-Only Model**: Input is a stream of edges.
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Mark and Erica are no longer friends.

Like · Add Friend
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Tyler and Cameron are friends with Mark.

Like · Add Friend
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Sean and Mark are now friends.

Like · Add Friend
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**Lawyers** are now friends with everyone.
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  ![Like · Add Friend](image)

• **Goal** Using small memory, compute properties of the graph.
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  Like · Add Friend

• **Goal** Using small memory, compute properties of the graph.

• All the earlier algorithms apply in insert-delete model:
Two Main Graph Stream Models

- **Insert-Only Model**: Input is a stream of edges.
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Using small memory, compute properties of the graph.

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- Maintain sketch $Mx$ where $x$ is characteristic vector of edges.
**Two Main Graph Stream Models**

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**Goal**

Using small memory, compute properties of the graph.

All the earlier algorithms apply in insert-delete model:

- Maintain sketch $M_x$ where $x$ is characteristic vector of edges.
- When $e$ inserted, update sketch $M_x \leftarrow M_x + (e^{th} \text{ column of } M)$
Part III

Streaming

Revisiting Matching
Correlation Clustering
Coloring Graphs
Coverage and Submodular Maximization
• **Unweighted Matching** Greedy algorithm returns 2-approx using $\tilde{O}(n)$ space. Embarrassingly, this is best known one-pass result!
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**Approximation Ratios for Weighted Matching**

<table>
<thead>
<tr>
<th>Feigenbaum et al.</th>
<th>McGregor</th>
<th>Zelke</th>
<th>Epstein et al.</th>
<th>Crouch-Stubbs</th>
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![Approximation Ratios for Weighted Matching](chart)

• **Weighted Matching** $2+\varepsilon$ approx in $\tilde{O}(n/\varepsilon)$ space.

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• **Weighted Matching** $2+\varepsilon$ approx in $\tilde{O}(n/\varepsilon)$ space. *Paz, Schwartzman [SODA 17]*

? Improve result for sparse graphs? Graph has *arboricity* $\alpha$ if all subgraphs have average degree $< \alpha$. Planar graph has $\alpha=3$. 
• **Thm** \( \alpha + 2 + \varepsilon \) approx of matching size in \( O(\text{polylog } n) \) space.

  Cormode et al. [ESA 17], McGregor, Vorotnikova [SOSA 18]
• **Thm** $\alpha + 2 + \varepsilon$ approx of matching size in $O(\text{polylog } n)$ space. 
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  b) Sample each edge w/p \( \approx \varepsilon^{-2} (\log n)/g \). If you subsequently see \( >\alpha \) edges incident to either endpoint, drop the edge.
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**Algorithm** Estimate $s$ up to a factor $1 + \varepsilon$

a) Suppose we have guess $g$ that is 2-approximates $s$

b) Sample each edge with probability $\approx \varepsilon^{-2} (\log n)/g$. If you subsequently see $> \alpha$ edges incident to either endpoint, drop the edge.

Can show a) the current sample size is always small and b) size of final sample and $g$ yields good approx for $s$. 
Part III

Streaming

Revisiting Matching

Correlation Clustering

Coloring Graphs

Coverage and Submodular Maximization
Consider a complete graph where edges are labelled attracted or repulsive. Given a node partition, an attracted edge is sad if it is cut and a repulsive edge is sad if it is not cut.
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  ![Graph Diagram]

• **Correlation Clustering** Find partition minimizing \# sad edges.

  See tutorial Bonchi, Garcia-Soriano, Liberty [KDD 14]
• Consider a complete graph where edges are labelled **attractive** or **repulsive**. Given a node partition, an attractive edge is sad if it is cut and a repulsive edge is sad if it is not cut.

**Correlation Clustering** Find partition minimizing # sad edges.

See tutorial Bonchi, Garcia-Soriano, Liberty [KDD 14]

• **3-Approx Algorithm**
  a) Pick random node.
  b) Form cluster with it and its attracted neighbors.
  c) Remove cluster from graph and repeat until nodes remain.

Ailon, Charikar, Newman [J.ACM 08]
• *Emulating algorithm in two passes:*
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• *Preprocess* Randomly order nodes, \( v_1, v_2, \ldots, v_n \).
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Ahn et al. [ICML 16]
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• With more work, can get \( \tilde{O}(n) \) space with \( O(\log \log n) \) passes. Can also find maximal independent sets.
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Streaming
Revisiting Matching
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Coloring Graphs
Coverage and Submodular Maximization
• **Coloring** With min number of colors, assign a color to every node such that no edge has monochromatic endpoints.
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  \text{Thm} \quad \text{Can color a graph in } \Delta+1 \text{ colors where } \Delta \text{ is max degree.}
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  ![Graph Diagram]

  - **Thm** Can color a graph in $\Delta+1$ colors where $\Delta$ is max degree.
  - How can we do this in a few passes with $\tilde{O}(n)$ space?
  - $O(\Delta \log \log n)$ passes via independent sets. Let’s do better!
\((1 + \varepsilon)\Delta\) Coloring

- a) Randomly color with \(\Delta/r\) colors.
- b) Store edges \(E'\) with monochromatic endpoints.
- c) Shade colors such that \(E'\) edges no longer monochromatic.

Bera, Ghosh [ArXiv 18]
• \((1+\varepsilon)\Delta\) Coloring
  a) Randomly color with \(\Delta/r\) colors.
  b) Store edges \(E'\) with monochromatic endpoints.
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*Bera, Ghosh* [ArXiv 18]
• *(1+\varepsilon)\Delta Coloring*  a) Randomly color with \Delta/r colors.  b) Store edges E' with monochromatic endpoints.  c) Shade colors such that E' edges no longer monochromatic.  

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• $(1+\varepsilon)\Delta$ Coloring  
  a) Randomly color with $\Delta/r$ colors. 
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• **Space Analysis**
  
\(|E'|=O(nr)\) since probability edge in \(E'\) is \(r/\Delta\).
(1+\varepsilon)\Delta Coloring

a) Randomly color with \Delta/r colors.

b) Store edges E’ with monochromatic endpoints.

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\[ \text{Bera, Ghosh [ArXiv 18]} \]

• **Space Analysis**  
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• **Colors Analysis**  
If r\approx \varepsilon^{-2} \log n, max degree in E’ is \Delta_{E’}< (1+\varepsilon)r and final number of colors is (1+\Delta_{E’})\Delta/r= (1+\varepsilon)\Delta.
(1+\varepsilon)\Delta Coloring  

a) Randomly color with \Delta/r colors.  
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Space Analysis  
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\Delta+1 Coloring Idea  
For node v, pick \(S_v \subset \{1, \ldots, \Delta+1\}\) of \(O(\log n)\) random colors. May assume v's color in \(S_v\).  

Assadi et al. [ArXiv 18]
Part III
Streaming
Revisiting Matching
Correlation Clustering
Coloring Graphs
Coverage and Submodular Maximization
• **Max-k-Coverage** Given a stream of subsets $S_1, \ldots, S_m$ of $[n]$, find $C$ that maximizes $f(C) = |\bigcup_{i \in C} S_i|$ subject to $|C| \leq k$. 
• **Max-k-Coverage** Given a stream of subsets $S_1, \ldots, S_m$ of $[n]$, find $C$ that maximizes $f(C)=|\bigcup_{i \in C} S_i|$ subject to $|C| \leq k$. 
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• **Submodular Functions** $f$ is sub-modular if for $A \subset B$ and $x \notin B$,

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$$
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\[
f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)
\]

• **Thm** $(1-\varepsilon)/2$ approx. of max-coverage in $\tilde{O}(\varepsilon^{-3}k)$ space.

*McGregor, Vu [ICDT 17]*
• **Algorithm** Guess $g$ such that $\text{OPT} \leq g \leq (1 + \varepsilon)\text{OPT}$. Add first $\leq k$ sets that each cover at least $g/(2k)$ new elements.
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• **Approx Ratio** If $k$ sets added, we cover $g/2 \geq \text{OPT}/2$. If less sets added, each set not added covers $<g/(2k)$ new elements and hence we covered $\text{OPT}-g/2 \geq \text{OPT}(1-\varepsilon)/2$. 
• **Algorithm** Guess \( g \) such that \( \text{OPT} \leq g \leq (1+\varepsilon)\text{OPT} \). Add first \( \leq k \) sets that each cover at least \( g/(2k) \) new elements.

• **Approx Ratio** If \( k \) sets added, we cover \( g/2 \geq \text{OPT}/2 \). If less sets added, each set not added covers \( <g/(2k) \) new elements and hence we covered \( \text{OPT}-g/2 \geq \text{OPT}(1-\varepsilon)/2 \).

• **Reducing Space** Above algorithm requires \( \tilde{O}(\varepsilon^{-1} \text{OPT}) \) space. Can use subsampling to such that \( \text{OPT} = \tilde{O}(\varepsilon^{-2} k) \).
• **Algorithm** Guess $g$ such that $\text{OPT} \leq g \leq (1+\varepsilon)\text{OPT}$. Add first $\leq k$ sets that each cover at least $g/(2k)$ new elements.

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• **Reducing Space** Above algorithm requires $\tilde{O}(\varepsilon^{-1} \text{OPT})$ space. Can use subsampling to such that $\text{OPT} = \tilde{O}(\varepsilon^{-2} k)$.

• **Generalizations** Constant passes for $\approx 1 - 1/e$ approx. Extends to other monotone submodular function. Other work on non-monotone functions, beyond cardinality constraints, etc.

  *McGregor, Vu [ICDT 17], Bateni et al. [SPAA 17], Assadi [PODS 17]*
Thanks! Over to Sudipto...